AN ELEMENTARY COURSE

IN

THEORY OF EQUATIONS.

BY

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NEW YORK:

JOHN WILEY & SONS,

53 EAST TENTH STREET.

1892.

ROBERT DRUMMOND,
Electrotyper,
444 & 446 Pearl Street,
New York.

FERRIS BROS.,
Printers,
326 Pearl Street,
New York.

PREFACE.

An elementary knowledge of Determinants and of the Theory of Equations is necessary to those beginning the study of modern higher mathematics, and this little book aims to give both a brief outline of these subjects and a working knowledge of those portions most frequently applied by the advanced student. It presupposes only a good knowledge of ordinary algebra, and perhaps a slight acquaintance with trigonometry and calculus, and is suitable either for a college text-book or for private study. While the exercises scattered through the book, together with the supplementary examples at the end, afford a good deal of practice for the student, it is hoped that teachers will stimulate the interest of their classes by inventing, often on the spur of the moment, suitable fresh illustrations and exercises.

The existing treatises on these subjects are, many of them, monuments of industry and learning, but are too exhaustive for the college student. The author has received great profit from several of them, especially from Burnside and Panton's *Theory of Equations;* but, as he has introduced only those theorems which have become classical in their departments, he has deemed it unnecessary to insert references to these treatises, or to the original memoirs.

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PART I.

DETERMINANTS.

1. Definition.—Let the n^2 quantities $a_1, a_2, \ldots a_n$; $b_1, b_2, \ldots b_n$; $\ldots l_n$ be arranged in n rows and columns as follows;

$$\begin{vmatrix} a_1 & a_2 & a_3 & \cdot & \cdot & \cdot & a_n \\ b_1 & b_2 & b_3 & \cdot & \cdot & \cdot & b_n \\ \cdot & \cdot & \cdot & & & & \\ \cdot & \cdot & \cdot & & & & \\ \cdot & \cdot & \cdot & & & & \\ l_1 & l_2 & l_3 & \cdot & \cdot & \cdot & l_n \end{vmatrix}.$$

If all possible products of n letters be formed by taking one and only one letter from each row and column, and if each product be affected with its proper sign, the algebraic sum of the products is called a determinant, and the square array is taken to denote this sum.

2. Rule of Signs.—The diagonal $a_1, b_2, \ldots l_n$ is called the principal diagonal of the determinant.

The product $a_1b_2c_3d_4 \ldots l_n$ of the letters in the principal diagonal is always affected with the sign $+$.

To determine the sign of any other product: first, put its letters in alphabetical order; then count the permutations necessary to bring the subscripts into the order 1 2 3 4 . . . of the subscripts in the principal diagonal. If they make an even number the term is affected with $+$; if they make an odd number the term is affected with $-$.

Suppose the principal diagonal product is $a_1b_2c_3d_4$; to find the sign of $a_3b_2c_4d_1$, we note that three permutations put the subscript 1 first; one permutation puts the 3 between 2 and 4; hence the whole number of permutations of subscripts to put them in the order 1 2 3 4 is four; consequently the sign of the term is $+$.

[Find the signs of $a_1b_3c_4d_2$; $b_4c_2a_1d_3$; $a_4b_2c_5d_1e_6$; $f_1b_2c_6e_3a_5d_4$, and many other products.]

Remark.—The order of a determinant is the number of its rows or columns. [Write down numerical determinants of orders 2, 3, 4, and form all their products with proper signs.] For instance:

$$\begin{vmatrix} 5 & 1 & 2 \\ 3 & 9 & 8 \\ 6 & 2 & 4 \end{vmatrix} = \begin{array}{l} 5\times 9\times 4 - 5\times 2\times 8 - 3\times 1\times 4 \\ +3\times 2\times 2 + 6\times 1\times 8 - 6\times 9\times 2. \end{array}$$

3. Theorem.—If the rows be made columns and the columns rows the value of the determinant is unchanged.

For the products can suffer no change but one of sign, and since the principal diagonal is quite unaltered there is no change of sign in any term.

[Form all the products for $\begin{vmatrix} 3 & 2 & 9 \\ 6 & 1 & 4 \\ 7 & 5 & 6 \end{vmatrix}$ and show that those of $\begin{vmatrix} 3 & 6 & 7 \\ 2 & 1 & 5 \\ 9 & 4 & 6 \end{vmatrix}$ are the same.]

4. Theorem.—Interchanging two adjacent rows changes the sign of every product. For, the diagonal term being originally $a_1b_2c_3 \dots l_n$, let the rows of b's and c's be interchanged. The diagonal term in the new determinant is $a_1c_2b_3 \dots$ or, in alphabetical order, $a_1b_3c_2d_4 \dots l_n$. This term must be affected with $+$ in the new determinant, while in the old one it was affected with $-$. The signs of all the terms in the new determinant are found by counting the permutations of suffixes necessary to put them in the form of the diagonal term $a_1b_3c_2 \dots l_n$; if the number is even they are affected with $+$; if odd, with $-$. But clearly if an even number of permutations of suffixes puts a term in the form $a_1b_3c_2 \dots l_n$, it will take an odd number to put it in the form $a_1b_2c_3 \dots l_n$; and if an odd number puts it in the form $a_1b_3c_2 \dots l_n$, it will take an even number to put it in the form $a_1b_2c_3 \dots l_n$. We conclude that negative terms in the new determinant are positive in the old one, and positive terms in the new determinant are negative in the old one. This proves the theorem. [The same theorem holds for columns. Prove it.]

5. Theorem.—If a row or column be moved past an odd number of rows or columns, the sign of the determinant changes; but not if it be moved past an even number of rows or columns. [Why?]

6. Theorem.—Interchanging any two rows or columns changes the sign of the determinant.

Let i and j be the numbers of the rows counting from the top; there are $j - i - 1$ rows between them. Now $j - i$ permutations put the jth row in the ith place, and $j - i - 1$ permutations more put the ith row in the vacant jth place. The number of permutations of rows necessary to make the interchange is therefore

$$j - i + j - i - 1 = 2(j - i) - 1,$$

which is always odd. Hence the theorem is true. [Why? Prove for columns.]

7. Theorem.—If two rows or columns have all their elements alike the determinant vanishes.

Interchanging the two rows must change the sign of the determinant; but if the two rows are the same, interchanging them alters nothing in the determinant. Calling the determinant D, we must then have $D = -D$, or $2D = 0$; hence $D = 0$. [Prove for columns. Show by forming all the products that $\begin{vmatrix} 1 & 3 & 4 \\ 2 & 5 & 6 \\ 1 & 3 & 4 \end{vmatrix} = 0$.]

8. Lemma.—The determinant

$$\begin{vmatrix} a_1 + x & b_1 + y & c_1 + z \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = H$$

is equal to

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} x & y & z \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = F + G.$$

For, a typical term of H is

$$-(a_1 + x)b_3c_2 = -a_1b_3c_2 - xb_3c_2;$$

of these terms the first comes from F and the second from G. In like manner it is seen that all the terms of F and G with their proper signs occur in H; and, moreover, that they make up the whole of H. The same is true for a determinant of any order. If each row of a determinant of the third order consists of sums of two terms, the determinant will break up into a sum of eight simpler ones. [Why?

If each row consists of sums of three terms, how many simpler determinants can be formed whose sum equals the original one? Write $\begin{vmatrix} 2+6+5, & 3+2+1, & 7+1+9 \\ 6+1+2, & 3+1+9, & 4+1+7 \\ 3+8+4, & 2+9+3, & 5+6+2 \end{vmatrix}$ as a sum of twenty-seven determinants.]

9. Theorem.—If all the elements of a row or column be multiplied by a common factor and added to the corresponding element of another row or column, the value of the determinant is not changed.

In $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ let us multiply the third row by λ and add it to the first row; we obtain

$$\begin{vmatrix} a_1+\lambda a_3 & b_1+\lambda b_3 & c_1+\lambda c_3 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

This breaks up into

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \lambda \begin{vmatrix} a_3 & b_3 & c_3 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

But the coefficient of λ has two rows alike and therefore vanishes; hence what we have done does not change the value of the determinant.

10. Theorem.—If each element in a row or column is multiplied or divided by a common factor, the determinant is multiplied or divided by that factor. For the factor will come once and only once into each product of n letters.

[Simplify $\begin{vmatrix} 2 & 6 & 12 \\ 9 & 27 & 15 \\ 4 & 16 & 32 \end{vmatrix}$ by removing common factors. Treat many other determinants in the same way.]

11. Definitions.—A first minor of a determinant of order n is formed by leaving out any row and any column; it is a determinant of order $n-1$. A second minor is formed by leaving out any two rows and columns; it is a determinant of order $n-2$. Similarly, we may form minors of any rank.

Suppose the determinant is $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$; if we leave out the first row and column we form the minor $\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix}$; by leaving out the third row and second column we form the minor $\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}$. The row and column omitted have one and only one letter in common; and the first minor is said to be taken with respect to this letter. Thus $\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix}$ is taken with respect to a_1; $\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}$ is taken with respect to c_2. The first minor taken with respect to a_1 is denoted by A_1; that corresponding to c_2 by C_2; and so on in general.

12. Signs of Minors.—A minor must always be affected with its proper sign. Let the letter with respect to which the first minor is taken stand in the ith row and jth column; the sign affecting the minor is $(-1)^{i+j}$. This rule is necessary in order to preserve the proper signs of the products of n letters. We have then

$$A_1 = \begin{vmatrix} b_2 & c_3 \\ c_2 & c_3 \end{vmatrix};\ A_2 = -\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix};\ C_2 = (-1)^{3+2}\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} = -\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}.$$

[Form all the first minors of $\begin{vmatrix} 6 & 2 & 1 & 7 \\ 3 & 8 & 4 & 6 \\ 2 & 2 & 3 & 9 \\ 5 & 1 & 4 & 3 \end{vmatrix}$ and give them their proper signs.]

13. Expansion by Minors.—The product a_1A_1 contains all the products of n letters of the original determinant in which a_1 appears, and with the correct signs; a_2A_2 contains all the products in which a_2 appears, and with the correct signs; continuing in this way, we see that $a_1A_1 + a_2A_2 + \ldots + a_nA_n$ contains all the products of n letters in which $a_1, a_2, \ldots a_n$ appear. But there is no product from which $a_1, a_2, \ldots a_n$ are all absent; hence $a_1A_1 + \ldots + a_nA_n$ contains all the products of n letters of the determinant of order n and with the correct signs; hence $a_1A_1 + \ldots + a_nA_n$ is the value of the determinant. Similarly the value of the determinant is $a_1A_1 + b_1B_1 + \ldots + l_1L_1$, where the minors are all taken with respect to letters standing in the same column and each minor multiplied by its own small letter.

[Expand $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ in all possible ways in terms of first minors.]

14. Theorem.—The value of $\begin{vmatrix} a_2 & b_2 & c_2 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ is 0; if we take the minors with respect to the letters of the first row, we find that they are the A_1, B_1, C_1 of $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$.

Hence the value of $\begin{vmatrix} a_2 & b_2 & c_2 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ is $a_2A_1 + b_2B_1 + c_2C = 0$.

Hence if the minors for the letters of any row or column be multiplied respectively by the corresponding letters of any other row or column, the sum of the products vanishes.

[Expand $\begin{vmatrix} 3 & 7 & 2 & 9 \\ 2 & 4 & 1 & 6 \\ 3 & 8 & 4 & 7 \\ 2 & 1 & 6 & 1 \end{vmatrix}$ in terms of its first minors; multiply the minors for the third row by the numbers of the second, and show that the result vanishes.]

15. Theorem.—If every element but one of a row or column vanishes, the order of the determinant can be lowered by unity. Suppose that a_t is not zero, while $a_1 = a_2 = a_3 = \ldots = a_n = 0$; the value of the determinant of order n is $a_1A_1 + a_2A_2 + \ldots + a_tA_t + \ldots + a_nA_n$, where every term vanishes except a_tA_t. Now A_t is a determinant of order $n-1$; hence the theorem is true. Thus the value of

$$\begin{vmatrix} 6 & 2 & 7 & 4 \\ 1 & 9 & 8 & 7 \\ 0 & 0 & 3 & 0 \\ 6 & 2 & 1 & 9 \end{vmatrix} \text{ is } (-1)^6 \times 3 \times \begin{vmatrix} 6 & 2 & 4 \\ 1 & 9 & 7 \\ 6 & 2 & 9 \end{vmatrix} = 3 \begin{vmatrix} 6 & 2 & 4 \\ 1 & 9 & 7 \\ 6 & 2 & 9 \end{vmatrix}.$$

[Reduce $\begin{vmatrix} 2 & 1 & 6 & 3 \\ 0 & 0 & 7 & 1 \\ 0 & 9 & 0 & 0 \\ 1 & 2 & 3 & 4 \end{vmatrix}$ and $\begin{vmatrix} 6 & 4 & 2 & 1 & 7 \\ 3 & 9 & 6 & 8 & 5 \\ 2 & 1 & 0 & 3 & 2 \\ 0 & 0 & 0 & 5 & 0 \\ 2 & 4 & 1 & 3 & 2 \end{vmatrix}$

to determinants of lower order.]

16. Rule for computing the Numerical Value of a Determinant.—By removing factors, or by subtracting multiples of rows or columns from a selected row or column, cause the number 1 to appear in a selected place. By aid of this 1 every other number in the row or column in which it stands

may be caused to vanish, and the order of the determinant diminished by unity. The new determinant may be treated in like manner, and so on till the order is low enough to admit of easy calculation.

17. **Example.**—Suppose we wish to calculate

$$\begin{vmatrix} 7 & 4 & 2 & 6 & 3 \\ 8 & 9 & 5 & 4 & 2 \\ 7 & 6 & 3 & 2 & 9 \\ 8 & 4 & 7 & 5 & 3 \\ 7 & 3 & 8 & 4 & 5 \end{vmatrix}.$$

Subtract the first row from each of the others:

$$\begin{vmatrix} 7 & 4 & 2 & 6 & 3 \\ 1 & 5 & 3 & -2 & -1 \\ 0 & 2 & 1 & -4 & 6 \\ 1 & 0 & 5 & -1 & 0 \\ 0 & -1 & 6 & -2 & 2 \end{vmatrix}.$$

Subtract 7 times the second row from the first; and the second row itself from the fourth:

$$\begin{vmatrix} 0 & -31 & -19 & 20 & 10 \\ 1 & 5 & 3 & -2 & -1 \\ 0 & 2 & 1 & -4 & 6 \\ 0 & -5 & 2 & 1 & 1 \\ 0 & -1 & 6 & -2 & 2 \end{vmatrix}.$$

Which reduces to

$$\begin{vmatrix} -31 & -19 & 20 & 10 \\ 2 & 1 & -4 & 6 \\ -5 & 2 & 1 & 1 \\ -1 & 6 & -2 & -2 \end{vmatrix}.$$

The fourth row may now be made to contain three zeros by combining the third column with each of the others: multiply it by 5 and add to the first; by 2 and subtract from the second; and subtract it unchanged from the fourth.

$$\begin{vmatrix} 69 & -59 & 20 & -10 \\ -18 & 9 & -4 & 10 \\ 0 & 0 & 1 & 0 \\ -11 & 10 & -2 & 0 \end{vmatrix}.$$

This reduces to $\begin{vmatrix} 69 & -59 & -10 \\ -18 & 9 & 10 \\ -11 & 10 & 0 \end{vmatrix}$,

Add the second row to the first: $\begin{vmatrix} 51 & -50 & 0 \\ -18 & 9 & 10 \\ -11 & 10 & 0 \end{vmatrix}$

$$= -10 \begin{vmatrix} 51 & -50 \\ -11 & 10 \end{vmatrix} = -10 \begin{vmatrix} 1 & -50 \\ -1 & 10 \end{vmatrix}$$

by adding the second column to the first. This equals

$$-10(10+50) = -600.$$

18. Remark.—A determinant of the third order can usually be easily computed by expanding it in minors.

Thus, $\begin{vmatrix} 7 & 6 & 3 \\ 6 & 2 & 9 \\ 8 & 4 & 5 \end{vmatrix}$

$$=7(10-36)-6(30-12)+8(54-6)=-182-108+384=94$$

19. Exercises.—Compute

$$\begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 6 & 10 & 15 \\ 1 & 4 & 10 & 20 & 35 \\ 1 & 5 & 15 & 35 & 70 \end{vmatrix}; \qquad \begin{vmatrix} 9 & 8 & 7 & 13 \\ 6 & 2 & 4 & 3 \\ 7 & 9 & 8 & 6 \\ 4 & 3 & 11 & 20 \end{vmatrix}.$$

Prove that

$$\begin{vmatrix} 0 & a & b \\ a & 0 & c \\ b & c & 0 \end{vmatrix} = 2abc; \quad \begin{vmatrix} (a+b)^2 & c^2 & c^2 \\ a^2 & (b+c)^2 & a^2 \\ b^2 & b^2 & (c+a)^2 \end{vmatrix} = 2abc(a+b+c)3.$$

[Invent and compute many numerical determinants.]

20. To Differentiate a Determinant.—Let all the terms of the n rows and columns be functions of x; denote the determinant by u; it is required to calculate $\frac{du}{dx}$.

We have by the calculus, if $u = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$,

$$u + du = \begin{vmatrix} a_1 + \frac{da_1}{dx}dx & a_2 + \frac{da_2}{dx}dx & a_3 + \frac{da_3}{dx}dx \\ b_1 + \frac{db_1}{dx}dx & b_2 + \frac{db_2}{dx}dx & b_3 + \frac{db_3}{dx}dx \\ c_1 + \frac{dc_1}{dx}dx & c_2 + \frac{dc_2}{dx}dx & c_3 + \frac{dc_3}{dx}dx \end{vmatrix}.$$

This breaks up into eight determinants; one of them contains the factor dx^3, and three more contain dx^2. These

are differentials of order higher than the first, and they vanish in the limit. Hence

$$u+du=u+dx\begin{vmatrix} a_1' & a_2 & a_3 \\ b_1' & b_2 & b_3 \\ c_1' & c_2 & c_3 \end{vmatrix}+dx\begin{vmatrix} a_1 & a_2' & a_3 \\ b_1 & b_2' & b_3 \\ c_1 & c_2' & c_3 \end{vmatrix}+dx\begin{vmatrix} a_1 & a_2 & a_3' \\ b_1 & b_2 & b_3' \\ c_1 & c_2 & c_3' \end{vmatrix},$$

where $a_1' = \dfrac{da_1}{dx}, \ldots$ Hence

$$\frac{du}{dx}=\begin{vmatrix} a_1' & a_2 & a_3 \\ b_1' & b_2 & b_3 \\ c_1' & c_2 & c_3 \end{vmatrix}+\begin{vmatrix} a_1 & a_2' & a_3 \\ b_1 & b_2' & b_3 \\ c_1 & c_2' & c_3 \end{vmatrix}+\begin{vmatrix} a_1 & a_2 & a_3' \\ b_1 & b_2 & b_3' \\ c_1 & c_2 & c_3' \end{vmatrix}.$$

Hence the derivative with respect to x of a determinant of order n consists of the sum of n determinants each of which differs from the original determinant by having the elements of one and only one column differentiated with respect to x.

If $u = \begin{vmatrix} ax^2 & b \\ c & dx^3 \end{vmatrix}$, $\dfrac{du}{dx} = \begin{vmatrix} 2ax & b \\ 0 & dx^3 \end{vmatrix} + \begin{vmatrix} ax^2 & 0 \\ c & 3dx^2 \end{vmatrix}$.

21. Exercises.—Differentiate $\begin{vmatrix} a_0 & a_1 & a_2 & a_3 \\ a_1 & 2a_2 & 3a_3 & 0 \\ 3a_0 & 2a_1 & a_2 & 0 \\ 0 & 3a_0 & 2a_1 & a_2 \end{vmatrix}$ with respect to a_1, a_2, a_3, a_0, respectively.

Differentiate with respect to t $\begin{vmatrix} \sin t & \cos t & \tan t \\ \sin 2t & \cos 2t & \tan 2t \\ \sin \dfrac{1}{t} & \cos \dfrac{1}{t} & \tan \dfrac{1}{t} \end{vmatrix}$.

22. Rule for Multiplication.—The determinant

$$\begin{vmatrix} a_1\alpha_1+a_2\alpha_2+a_3\alpha_3 & a_1\beta_1+a_2\beta_2+a_3\beta_3 & a_1\gamma_1+a_2\gamma_2+a_3\gamma_3 \\ b_1\alpha_1+b_2\alpha_2+b_3\alpha_3 & b_1\beta_1+b_2\beta_2+b_3\beta_3 & b_1\gamma_1+b_2\gamma_2+b_3\gamma_3 \\ c_1\alpha_1+c_2\alpha_2+c_3\alpha_3 & c_1\beta_1+c_2\beta_2+c_3\beta_3 & c_1\gamma_1+c_2\gamma_2+c_3\gamma_3 \end{vmatrix},$$

which we shall denote by $\varDelta$, breaks up into the sum of twenty-seven determinants, of which twenty-one have two or more columns identical and therefore vanish. The remaining six are of the type

$$\begin{vmatrix} a_1\alpha_1 & a_3\beta_3 & a_2\gamma_2 \\ b_1\alpha_1 & b_3\beta_3 & b_2\gamma_2 \\ c_1\alpha_1 & c_3\beta_3 & c_2\gamma_2 \end{vmatrix} = \alpha_1\beta_3\gamma_2 \begin{vmatrix} a_1 & a_3 & a_2 \\ b_1 & b_3 & b_2 \\ c_1 & c_3 & c_2 \end{vmatrix} = -\alpha_1\beta_3\gamma_2 \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

Each of them contains $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$ as a factor, and their sum is $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$ multiplied by the factor

$$(\alpha_1\beta_2\gamma_3 - \alpha_1\gamma_2\beta_3 - \alpha_2\beta_1\gamma_3 + \alpha_2\gamma_1\beta_3 + \alpha_3\beta_1\gamma_2 - \alpha_3\beta_2\gamma_1).$$

The second factor is the determinant $\begin{vmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{vmatrix}$.

Hence $\varDelta = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \times \begin{vmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{vmatrix} = D \times D_1$, say.

The first row in $\varDelta$ is formed by taking for the first term the sum of the products of the first row of D into the first row of D_1; for the second term, the first row of D into the second row of D_1; and so on. To form the second row in $\varDelta$, the second row of D is multiplied into the first, second,

and third rows of D_1; and so on. We might just as well form Δ by multiplying the rows of D into the columns of D_1, or the columns of D into the rows of D_1, or the columns of D into the columns of D_1. The results would differ in form, but not in value. The same rule holds for two determinants of any order; but the two factors must be always of the same order. The product is a determinant of the same order as the factors.

23. Example.—Let us find the product of $\begin{vmatrix} 2 & 1 & 3 & 4 \\ 3 & 2 & 6 & 1 \\ 4 & 4 & 1 & 1 \\ 2 & 2 & 3 & 3 \end{vmatrix}$ and $\begin{vmatrix} 1 & 1 & 1 & 1 \\ 6 & 3 & 2 & 4 \\ 2 & 1 & 3 & 1 \\ 3 & 4 & 1 & 2 \end{vmatrix}$, taking rows by rows. The first term of the first row of the product is

$$2 \times 1 + 1 \times 1 + 3 \times 1 + 4 \times 1 = 10;$$

the second term of the first row is

$$2 \times 6 + 1 \times 3 + 3 \times 2 + 4 \times 4 = 37;$$

and so on. The first term in the second row of the product is

$$3 \times 1 + 2 \times 1 + 6 \times 1 + 1 \times 1 = 12;$$

the second term is

$$3 \times 6 + 2 \times 3 + 6 \times 2 + 1 \times 4 = 40;$$

and so on. The entire product is

$$\begin{vmatrix} 10 & 37 & 18 & 21 \\ 12 & 40 & 27 & 25 \\ 10 & 42 & 16 & 31 \\ 10 & 36 & 18 & 23 \end{vmatrix}.$$

[Find the product of $\begin{vmatrix} 2 & 1 & 6 \\ 9 & 2 & 8 \\ 3 & 1 & 3 \end{vmatrix}$ by $\begin{vmatrix} 4 & 1 & 4 \\ 2 & 1 & 2 \\ 2 & 4 & 1 \end{vmatrix}$; and of many other numerical determinants.]

24. Definition.—A determinant containing an unknown quantity and equated to 0 forms a determinant equation.

$\begin{vmatrix} a & x \\ x & b \end{vmatrix} = 0$ is a determinant equation. Written out it is $ab - x^2 = 0$, whence $x = \pm \sqrt{ab}$. The equation

$$\begin{vmatrix} a - \lambda, & h, & g \\ h, & b - \lambda, & f \\ g, & f, & c - \lambda \end{vmatrix} = 0,$$

where λ is the unknown quantity, is of great importance in geometry. Its roots are always real, provided that a, h, g, etc., are real quantities.

25. Solution of Linear Equations.—Given the equations

$$\begin{aligned} a_1x + a_2y + a_3z &= a, \\ b_1x + b_2y + b_3z &= b, \\ c_1x + c_2y + c_3z &= c. \end{aligned}$$

Let us form the determinant $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = D$, and take the first minors A_1, B_1, C_1 with respect to the first column. Multiply both members of the first equation by A_1, of the second by B_1, of the third by C_1, and add the results. The coefficient of x in the sum is $a_1A_1 + b_1B_1 + c_1C_1 = D$.

The coefficient of y is $a_2A_1 + b_2B_1 + c_2C_1 = 0$; and the coefficient of z is $a_3A_1 + b_3B_1 + c_3C_1 = 0$; hence the sum is

$$Dx = aA_1 + bB_1 + cC_1.$$

Now the minors of $\begin{vmatrix} a & a_2 & a_3 \\ b & b_2 & b_3 \\ c & c_2 & c_3 \end{vmatrix}$ with respect to a, b, c, are A_1, B_1, C_1; hence $aA_1 + bB_1 + cC_1 = \begin{vmatrix} a & a_2 & a_3 \\ b & b_2 & b_3 \\ c & c_2 & c_3 \end{vmatrix}$, and finally

$$Dx = \begin{vmatrix} a & a_2 & a_3 \\ b & b_2 & b_3 \\ c & c_2 & c_3 \end{vmatrix}.$$

If we multiply the equations by A_2, B_2, C_2, we shall find

$$Dy = aA_2 + bB_2 + cC_2 = \begin{vmatrix} a_1 & a & a_3 \\ b_1 & b & b_3 \\ c_1 & c & c_3 \end{vmatrix};$$

and finally by multiplying by A_3, B_3, C_3,

$$Dz = aA_3 + bB_3 + cC_3 = \begin{vmatrix} a_1 & a_2 & a \\ b_1 & b_2 & b \\ c_1 & c_2 & c \end{vmatrix}.$$

D is called the determinant of the left members. The rule for solving linear equations, as we see now, is:

Dx is equal to D with its first column replaced by the known quantities a, b, c; Dy is equal to D with its second column replaced by a, b, c; and Dz is equal to D with its third column replaced by a, b, c.

26. Example.—Suppose we are to solve

$$\begin{aligned} 3x + 2y - 4z &= 6, \\ 2x - 9y + 11z &= 4, \\ x - y - z &= 1. \end{aligned}$$

Here $D = \begin{vmatrix} 3 & 2 & -4 \\ 2 & -9 & 11 \\ 1 & -1 & -1 \end{vmatrix} = 58$; hence

$$58x = \begin{vmatrix} 6 & 2 & -4 \\ 4 & -9 & 11 \\ 1 & -1 & -1 \end{vmatrix},$$

$$58y = \begin{vmatrix} 3 & 6 & -4 \\ 2 & 4 & 11 \\ 1 & 1 & -1 \end{vmatrix},$$

$$58z = \begin{vmatrix} 3 & 2 & 6 \\ 2 & -9 & 4 \\ 1 & -1 & 1 \end{vmatrix}.$$

[In a similar way solve

$$\begin{aligned} 8x - y + 2z - 3w &= 1; \\ 4x - 3y - z + w &= -1; \\ x + 2y + z - 2w &= 2; \\ -x - y + z + w &= 3; \end{aligned}$$

and several other systems in three or four unknown quantities.]

27. Homogeneous Linear Equations.—A homogeneous linear equation in x, y, z is of the form $ax + by + cz = 0$, there being no absolute term. Such an equation contains really only two unknown quantities, $\frac{x}{z}$ and $\frac{y}{z}$; hence we can in general satisfy only two homogeneous linear equations with the three variables x, y, z. If then there be three homogeneous linear equations in x, y, z, either they can-

not all be satisfied by the same values of $\frac{x}{z}$, $\frac{y}{z}$, or the third equation must be a consequence of the first two. Let

$$\begin{aligned} a_1x + b_1y + c_1z &= 0, \\ a_2x + b_2y + c_2z &= 0, \\ a_3x + b_3y + c_3z &= 0, \end{aligned}$$

be the three equations, homogeneous and linear, in x, y, z. Let $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$, the determinant of the first members. From the equations

$$\begin{aligned} a_1x + b_1y + c_1z &= 0, \\ a_2x + b_2y + c_2z &= 0, \end{aligned}$$

we find $x : y : z :: b_1c_2 - b_2c_1 : c_1a_2 - a_1c_2 : a_1b_2 - b_1a_2$; but we observe upon examining D that these quantities are A_3, B_3, C_3. Now, if the three equations are consistent, these values of x, y, z must satisfy the third one; that is, we must have $a_3A_3 + b_3B_3 + c_3C_3 = 0$; but $a_3A_3 + b_3 + B_3$ $c_3C_3 = D$; hence the condition that the equations be consistent is $D = 0$. In general if n linear homogeneous equations in n variables are consistent their determinant must vanish; if the determinant does not vanish, the only set of values which will satisfy all the equations is $x = y = z = \ldots = 0$.

[Determine k so that the equations $3x + 2y + 4z = 0$, $9x + ky + 11z = 0$, $x + y + z = 0$, may be consistent. What do we mean by three equations being consistent? How many non-homogeneous equations can be satisfied by x, y, z, t simultaneously? How many independent homogeneous equations? Why?]

28. Remark.—It may also be noted that n non-homogeneous linear equations in n unknown quantities cannot be solved if the determinant of their left members vanishes.

29. Elimination by Sylvester's Dialytic Method.—The determinant D of the left members of n homogeneous linear equations is called their resultant. We can find the resultant of two non-homogeneous equations of the forms

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_{n-1}x + a_n = 0,$$
$$b_0x^m + b_1x^{m-1} + b_2x^{m-2} + \ldots + b_{m-1}x + b_m = 0,$$

where m and n are any two positive whole numbers, as follows: Multiply the first equation in succession by x^{m-1}, $x^{m-2}, \ldots x, x^0$; and the second equation by $x^{n-1}, x^{n-2}, \ldots$ x, x^0, successively. We thus form $m + n$ equations which may be regarded as homogeneous in the $m + n$ unknown quantities $x^{m+n}, x^{m+n-1}, \ldots x, x^0$, and if they are consistent the determinant of their left members must vanish. This determinant is

$$\begin{vmatrix}
a_0 & a_1 & a_2 \ldots a_n & 0 & 0 & 0 \ldots 0 \\
0 & a_0 & a_1 \ldots a_{n-1} & a_n & 0 & 0 \ldots 0 \\
0 & 0 & a_0 \ldots a_{n-2} & a_{n-1} & a_n & 0 \ldots 0 \\
\multicolumn{6}{l}{\text{There are } m \text{ rows formed in this way.}} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \quad \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \quad \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \quad \cdot \\
b_0 & b_1 & b_2 \ldots\ldots\ldots\ldots & & & \ldots 0 \\
0 & b_0 & b_1 \ldots\ldots\ldots\ldots & & & \ldots 0 \\
0 & 0 & b_0 \ldots\ldots\ldots\ldots & & & \ldots 0 \\
\multicolumn{6}{l}{\text{There are } n \text{ rows formed in this way.}}
\end{vmatrix}.$$

This determinant is the resultant of the two equations; if it vanishes they have a common root. This method of

eliminating the powers of x from the two equations is called Sylvester's Dialytic Method of Elimination.

30. Examples.—The resultant of $3x^3 - 7x^2 - 8x + 1 = 0$ and $2x^2 - x + 3 = 0$ is $\begin{vmatrix} 3 & -7 & -8 & 1 & 0 \\ 0 & 3 & -7 & -8 & 1 \\ 2 & -1 & 3 & 0 & 0 \\ 0 & 2 & -1 & 3 & 0 \\ 0 & 0 & 2 & -1 & 3 \end{vmatrix}$; have they a common root?

Eliminate x from $x^4 + 4x^3 + 12x^2 + 24x + 24 = 0$ and $x^3 + 3x^2 + 6x + 6 = 0$, by the dialytic method. The coefficients of the first must be written three times, those of the second four times.

31. Equation of a Straight Line.—Let $Ax + By + C = 0$ be the equation of a straight line. If it passes through the points (x_1, y_1) and (x_2, y_2), we must have $Ax_1 + By_1 + C = 0$ and $Ax_2 + By_2 + C = 0$. Eliminating A, B, C from these three equations, we find $\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$ for the determinant form of the equation of a line through two points.

[Form the equations of the lines through several pairs of points.]

32. Area of a Triangle.—Let (a, b), (a_1, b_1), (a_2, b_2) be the rectangular co-ordinates of the vertices of a plane triangle. The equation of the side opposite (a, b) is

$$\begin{vmatrix} x & y & 1 \\ a_1 & b_1 & 1 \\ a_2 & b_2 & 1 \end{vmatrix} = 0.$$

In its normal form this equation is

$$\frac{1}{\sqrt{(b_1 - b_2)^2 + (a_1 - a_2)^2}} \begin{vmatrix} x & y & 1 \\ a_1 & b_1 & 1 \\ a_2 & b_2 & 1 \end{vmatrix} = 0.$$

The perpendicular distance from (a, b) upon this side is therefore $\dfrac{1}{\sqrt{(a_1 - a_2)^2 + (b_1 - b_2)^2}} \begin{vmatrix} a & b & 1 \\ a_1 & b_1 & 1 \\ a_2 & b_2 & 1 \end{vmatrix}$.

The area of the triangle is half the product of this perpendicular into the length of the side (a_1, b_1), (a_2, b_2); this length is $\sqrt{(a_1 - a_2)^2 + (b_1 - b_2)^2}$; hence the area of the triangle is $\frac{1}{2} \begin{vmatrix} a & b & 1 \\ a_1 & b_1 & 1 \\ a_2 & b_2 & 1 \end{vmatrix}$.

33. Examples.—The area of the triangle whose vertices are $(7, 5)$, $(2, -6)$, $(1, 1)$ is $\begin{vmatrix} 7 & 5 & 1 \\ 2 & -6 & 1 \\ 1 & 1 & 1 \end{vmatrix}$. Find the area of the triangle whose vertices are $(-1, 0)$, $(2, 5)$, $(3, -4)$; and of several others.

PART II.

ALGEBRAIC EQUATIONS.

1. Definition.—A polynomial in x containing only positive integral powers of x, and whose coefficients are either real whole numbers or real rational fractions, forms, when equated to 0, an algebraic equation.

If the coefficients are letters, the equation is literal; otherwise it is numerical.

An example is

$$1) \qquad a_0x^n + a_1x^{n-1} + \ldots + a_{n-1}x + a_n = 0,$$

where n is a positive whole number and $a_0, a_1, \ldots$ are real, rational numbers. We may denote the left-hand member of (1) by $f(x)$, and shall say that $f(x)$ is a rational, integral, algebraic function of x, of degree n.

[Write several such functions with letters and numbers.]

2. Roots.—In $f(x)$, x is called the variable; and the coefficients of its different powers are constant quantities. Since any value of x substituted in $f(x)$ will give some value to the function, we may write $f(x) = y$, and then y will have one, and only one, value for each value of x. For certain values of $x, f(x)$ reduces to zero; such values of x are called the roots of the equation $f(x) = 0$. To *solve* a numerical equation *is to find its roots.* This can always be done either exactly, or as nearly as we choose by methods of approximation.

3. Theorem.—$f(x)$ is a continuous function of x; that is, for any two values of x which differ by a small quantity of the first order, the two corresponding values of $f(x)$ can differ only by a small quantity of the first order.

For let x and $x + h$ be two consecutive values of x, h being small; then by Taylor's Theorem

$$2) \qquad f(x+h) = f(x) + hf'(x) + \frac{h^2}{1.2}f''(x) + \ldots;$$

the omitted terms all containing powers of h higher than the second. We have then

$$f(x+h) - f(x) = hf'(x) + h^2(P),$$

where

$$P = \frac{f''(x)}{2!} + \frac{h}{3!}f'''(x) + \frac{h^2}{4!}f^{\text{iv}}(x) + \ldots + \frac{h^{n-2}}{n!}a_0,$$

and is therefore finite for all finite values of x. Hence h^2P is a small quantity of the second order, and we conclude that $f(x + h)$ differs from $f(x)$ only by a small quantity of the first order.

4. $f(x)$ describes a Curve.—Hence if we write $y = f(x)$, and give to x a consecutive series of values, the values of y will be consecutive; so that if x be the abscissa, y will be the ordinate of a curve. Thus taking

$$y = 10x^3 - 17x^2 + x + 6,$$

we have for

$x =$	-4	-3	-2	-1	0	1	2	3	4
$y =$	-910	-420	-144	-22	6	0	20	126	378

5. Roots.—At points where the curve cuts the axis of x, $y = 0$; hence the values of x at such points are real roots

of $f(x) = 0$. It is important to note that as x passes a real root of $f(x) = 0$, y changes sign. If the curve cuts the axis of x n times, the roots are all real; if not, some of the roots are imaginary.

6. To Compute $f(h)$.—There is an easy way to compute the value of $f(x)$ when x has a given value. It is as follows: Let h be any quantity, and let $f(x)$ when divided by $x - h$ give a quotient Q and a remainder R; then shall

$$3) \qquad f(x) = (x - h)Q + R$$

identically; that is, the various powers of x on both sides of the equation have the same coefficients. Q cannot be of higher degree than $n - 1$; let it be the polynomial

$$b_0x^{n-1} + b_1x^{n-2} + \dots + b_{n-2}x + b_{n-1}.$$

Then $(x - h)Q + R =$

$$\begin{array}{lr|lr|lr|l} & b_1 & x^{n-1} + & b_2 & x^{n-2} + \dots + & b_{n-1} & x + R \\ b_0x^n & -hb_0 & & -hb_1 & & -hb_{n-2} & -hb_{n-1} \end{array}$$

Remembering that

$$f(x) = a_0x^n + a_1x^{n-1} + \dots + a_{n-1}x + a_n,$$

we must have

$$\begin{aligned} b_0 &= a_0, \\ b_1 &= b_0h + a_1, \\ b_2 &= b_1h + a_2, \\ &\cdot \\ &\cdot \\ &\cdot \\ b_{n-1} &= b_{n-2}h + a_{n-1}, \\ R &= b_{n-1}h + a_n. \end{aligned}$$

Arranging these quantities in the following way:

$$\begin{array}{cccccc} a_0 & a_1 & a_2 & a_3 \; \ldots & a_{n-1} & a_n \\ & b_0h & b_1h & b_2h \; \ldots & b_{n-2}h & b_{n-1}h \\ \hline b_0 & b_1 & b_2 & b_3 & b_{n-1} & R \end{array}$$

we easily compute $b_0, b_1, \ldots b_{n-1}$, the coefficients of Q, and finally R, the remainder. We start with b_0 which $= a_0$. Multiply this by h and add to a_1—it gives b_1; multiply this by h and add to a_2—it gives b_2; and so on to the end. Finally multiply b_{n-1} by h and add to a_n, and we shall have R. Now R is the result obtained by making $x = h$ in $f(x)$; for in equation (3) make $x = h$; we have then

$$4) \qquad f(h) = R.$$

[Find quotients and remainders. Tabulate values of functions for various values of x. Plot curves.]

7. The Quantity $\sqrt{-1}$.—The quantity $\sqrt{-1}$ is generally denoted by i; it satisfies the equation $i^2 = -1$.

It is a constant quantity; much more it cannot vanish; hence in any such equation as $yi = 0$ we must have $y = 0$.

8. Definitions.—A quantity whose square is positive is a real quantity. Multiples of i are called pure imaginaries: their squares are negative.

9. Definition.—Let x and y be real; then $x + iy$ is called a complex quantity, being the sum of a real and an imaginary quantity.

10. Theorem.—If $x + iy = 0$, then must $x = 0$, $y = 0$. Otherwise we should have $x = -iy$; but x is real by hypothesis.

11. Theorem.—If $x + iy = a + ib$, then $x = a$, $y = b$: otherwise we should have $x - a = i(b - y)$; but by hypothesis $x - a$ is real.

12. Theorem.—The algebraic sum of any number of complex quantities is a complex quantity. For

$$x + yi \pm (x' + y'i) = (x \pm x') + (y \pm y')i.$$

13. Theorem.—The product of two complex quantities is a complex quantity. For

$$(x + yi)(a + ib) = ax - by + i(bx + ay).$$

It follows that the product of any number of complex quantities is a complex quantity.

14. Theorem.—The quotient of two complex quantities is a complex quantity. For

$$\frac{x + iy}{a + ib} = \frac{(x + iy)(a - ib)}{a^2 + b^2} = \frac{ax + by + i(ay - bx)}{a^2 + b^2}.$$

15. Definition and Theorem.—A rational function is one in forming which no operations are performed but addition, subtraction, multiplication, and division; and since these operations performed on complex quantities can only reproduce complex quantities, it follows that every rational function of complex quantities is itself a complex quantity and may be put in the form $P + iQ$.

16. Form of $f(x - iy)$.—Let $f(x + iy)$ be not only rational but also integral; let its coefficient be real, and finally let

$$f(x + iy) = P + iQ.$$

Then we see that no odd power of y can be contained in P, for every odd power of y goes with an i factor; on the other hand, no even power of y can be contained in Q, for no even power of y goes with an i factor. Moreover if Q

be the algebraic sum of several terms, every one of them must contain an odd power of y, for in no other way could they be made to go with an i factor. Since odd powers of y change their sign when y changes from $+y$ to $-y$, but even powers of y remain unaffected, we conclude that

$$f(x - iy) = P - iQ.$$

17. Definition.—The complex quantities $a + ib$ and $a - ib$ are called conjugate imaginaries. Their product, $a^2 + b^2$, is real.

18. Theorem.—If $a + ib$ is a root of an algebraic equation, then also is $a - ib$ a root of the same equation.

For, let $f(x) = 0$ be the equation. If $a + ib$ is a root, we must have $f(a + ib) = 0$. But this may be written

$f(a + ib) = P + iQ = 0$; and this requires $P = 0$, $Q = 0$.

Hence $iQ = 0$, and finally $P - iQ = 0$. Now

$$P - iQ = f(a - ib);$$

hence $a - ib$ is a root of $f(x)$.

19. Remark. Definitions.—The sum of the conjugate imaginaries $a + ib$, $a - ib$, is the real quantity $2a$; their difference is the pure imaginary $2ib$. Their product $a^2 + b^2$ is called the *Norm* of either of them.

$$\text{Thus norm } (a + ib) = \text{norm } (a - ib) = a^2 + b^2.$$

The positive square root of the norm is the modulus. This quantity is indicated by the following symbol:

$$\text{mod } (a + ib) = \sqrt{a^2 + b^2};$$
$$\text{mod } (a - ib) = \sqrt{a^2 + b^2}.$$

20. Remark.—A quantity and its conjugate imaginary have the same modulus.

[Compute the moduli of several quantities.]

21. Remark.—When $y = 0$, we notice that

$$\text{mod } (x + iy) = + \sqrt{x^2} = + x;$$

that is, it is the numerical magnitude of x, or its absolute value; hence for instance,

$$\text{mod } (-3) = +3, \quad \text{mod } (\sqrt{16}) = +4.$$

22. Theorem.—If $x + iy = 0$, then also mod $(x + iy) = 0$. For $x + iy = 0$ implies $x = 0$, $y = 0$; hence $\sqrt{x^2 + y^2} = 0$.

23. The Converse.—If mod $(x + iy) = 0$, then also $x + iy = 0$. For $x^2 + y^2 = 0$ implies $x = 0$, $y = 0$.

24. Mod $f(x \pm iy)$.—If $f(x + iy) = P + iQ$, then

$$f(x - iy) = P - iQ \text{ and } f(x + iy)\, f(x - iy) = P^2 + Q^2;$$

so that $\quad \text{mod } f(x + iy) = \sqrt{P^2 + Q^2} = \text{mod } f(x - iy).$

25. Theorems.—The modulus of the product of two complex members is the product of their moduli. For

$$(a + ib)(x + iy) = ax - by + i(bx + ay).$$

Hence
$$\begin{aligned} \text{mod } (a + ib)(x + iy) &= \sqrt{(ax - by)^2 + (bx + ay)^2} \\ &= \sqrt{(a^2 + b^2)(x^2 + y^2)}. \end{aligned}$$

The modulus of their quotient is the quotient of their moduli. For

$$\frac{a + ib}{x + iy} = \frac{(a + ib)(x - iy)}{x^2 + y^2}.$$

Hence

$$\operatorname{mod}\frac{a+ib}{x+iy}=\operatorname{mod}\frac{1}{x^2+y^2}(a+ib)(x-iy)$$
$$=\frac{\sqrt{(a^2+b^2)(x^2+y^2)}}{x^2+y^2}=\sqrt{\frac{a^2+b^2}{x^2+y^2}}.$$

26. Theorem.—The modulus of the sum of two complex quantities cannot exceed the sum of their moduli.

Let $x+iy$ and $a+ib$ be the quantities. The sum of their moduli is $\sqrt{x^2+y^2}+\sqrt{a^2+b^2}$. The modulus of their sum is $\sqrt{(x+a)^2+(y+b)^2}$; but always

$$\sqrt{(x+a)^2+(y+b)^2}\overset{=}{<}\sqrt{x^2+y^2}+\sqrt{a^2+b^2}.$$

For.

$$(x+a)^2+(y+b)^2\overset{=}{<}x^2+y^2+a^2+b^2+2\sqrt{x^2+y^2}\sqrt{a^2+b^2};$$

that is,

$$2ax+2by\overset{=}{<}2\sqrt{x^2+y^2}\sqrt{a^2+b^2},$$

or

$$a^2x^2+b^2y^2+2abxy\overset{=}{<}(x^2+y^2)(a^2+b^2),$$

or

$$2abxy\overset{=}{<}b^2x^2+a^2y^2,$$

or

$$(bx-ay)^2\overset{=}{>}0, \text{ which is true.}$$

If $bx-ay=0$, then $\dfrac{a}{x}=\dfrac{b}{y}=\lambda$, say; and $x+iy$ is a multiple of $a+ib$.

This theorem may be extended by similar reasoning to include any number of quantities; hence the sum of the moduli of any number of complex quantities can never be less than the modulus of their sum, and is generally greater.

27. The Trigonometric Form of $x + iy$.—Let us now write the complex quantity in the form

$$x + iy = +\sqrt{x^2 + y^2}\left(\frac{x}{+\sqrt{x^2 + y^2}} + i\frac{y}{+\sqrt{x^2 + y^2}}\right),$$

and observe that

$$\frac{x^2}{x^2 + y^2} + \frac{y^2}{x^2 + y^2} = 1$$

Hence there is an angle ϑ such that

$$\frac{x}{\sqrt{x^2 + y^2}} = \cos\vartheta, \quad \frac{y}{\sqrt{x^2 + y^2}} = \sin\vartheta, \quad \text{and} \quad \tan\vartheta = \frac{x}{y}.$$

If we denote $+\sqrt{x^2 + y^2}$ by r, then we shall have

$$x + iy = r(\cos\vartheta + i\sin\vartheta).$$

Here r is the modulus of $x + iy$, and the angle ϑ is called the argument of $x + iy$.

[Determine r and ϑ for several quantities.]

28. The Exponential Form of $x + iy$.—We have then the complex quantity $x + iy$ the relations

$$\frac{x}{r} = \cos\vartheta, \quad \frac{y}{r} = \sin\vartheta, \quad \text{and hence} \quad \frac{y}{x} = \tan\vartheta.$$

Now we know that we may write $\cos\vartheta$ in the form

$$\cos\vartheta = 1 - \frac{\vartheta^2}{2!} + \frac{\vartheta^4}{4!} - \frac{\vartheta^6}{6!} + \dots,$$

and $\sin\vartheta$ in the form

$$\sin\vartheta = \vartheta - \frac{\vartheta^3}{3!} + \frac{\vartheta^5}{5!} - \frac{\vartheta^7}{7!} + \dots;$$

whence

$$\cos\vartheta + i\sin\vartheta = 1 + i\vartheta - \frac{\vartheta^2}{2!} - i\frac{\vartheta^3}{3!} + \frac{\vartheta^4}{4!} + i\frac{\vartheta^5}{5!}\dots.$$

If now we define a function $e^{i\vartheta}$ by the series

$$e^{i\vartheta} = 1 + i\vartheta + \frac{(i\vartheta)^2}{2!} + \frac{(i\vartheta)^3}{3!} + \frac{(i\vartheta)^4}{4!} + \frac{(i\vartheta)^5}{5!} + \dots$$

$$= 1 + i\vartheta - \frac{\vartheta^2}{2!} - i\frac{\vartheta^3}{3!} + \frac{\vartheta^4}{4!} + i\frac{\vartheta^5}{5!}\dots,$$

which is entirely analogous to the form for e^x, where x is real, found by elementary calculus; then we notice that

$$e^{i\vartheta} = \cos\vartheta + i\sin\vartheta,$$

and consequently

$$x + iy = r(\cos\vartheta + i\sin\vartheta) = re^{i\vartheta}.$$

[Express complex quantities as powers of e, always expressing ϑ in angular measure, not in degrees.]

29. Properties of $e^{i\vartheta}$.—Again, take

$$a + ib = m(\cos\alpha + i\sin\alpha) = me^{i\alpha}.$$

Then

$$(x+iy)(a+ib) = mr(\cos\vartheta + i\sin\vartheta)(\cos\alpha + i\sin\alpha)$$
$$= mr\{\cos\vartheta\cos\alpha - \sin\vartheta\sin\alpha + i(\sin\alpha\cos\vartheta + \sin\vartheta\cos\alpha)\}$$
$$= mr\{\cos(\alpha+\vartheta) + i\sin(\alpha+\vartheta)\}.$$

But also

$$(x+iy)(a+ib) = re^{i\vartheta}me^{ia} = mre^{i\vartheta}e^{ia}.$$

We conclude that

$$e^{i\vartheta} \,.\, e^{ia} = \cos(\alpha+\vartheta) + i\sin(\alpha+\vartheta).$$

The second member defines the symbol $e^{i(\alpha+\vartheta)}$; henc finally

$$e^{i\vartheta}e^{ia} = e^{i(\alpha+\vartheta)}.$$

Again,

$$\frac{x+iy}{a+ib} = \frac{r}{m}\cdot\frac{\cos\vartheta + i\sin\vartheta}{\cos\alpha + i\sin\alpha} = \frac{r}{m}\,\frac{\{\cos\vartheta + i\sin\vartheta\}\{\cos\alpha - i\sin\alpha\}}{\cos^2\alpha + \sin^2\alpha}$$
$$= \frac{r}{m}\{\cos\vartheta\cos\alpha + \sin\vartheta\sin\alpha + i(\sin\vartheta\cos\alpha - \cos\vartheta\sin\alpha)\}$$
$$= \frac{r}{m}\{\cos(\vartheta-\alpha) + i\sin(\vartheta-\alpha)\} = \frac{r}{m}e^{i(\vartheta-\alpha)}$$

by definition. But $\dfrac{x+iy}{a+ib} = \dfrac{r}{m}\,\dfrac{e^{i\vartheta}}{e^{ia}}$. We conclude that

$$\frac{e^{i\vartheta}}{e^{ia}} = e^{i(\vartheta-\alpha)}.$$

The function $e^{i\theta}$ which we have defined obeys therefore the same laws of multiplication and division as the function e^x where x is real.

[Express products and quotients of complex quantities as powers of e.]

30. Argand's Diagram.—We shall now agree to lay off pure imaginaries along the axis of y, and real quantities along the axis of x; the axes being rectangular. To lay off ib, take a length equal to b on the axis of y. The quantity $a + ib$ will thus locate a point P in the plane of the axes, whose rectangular co-ordinates are a and b. The distance $OP = \sqrt{a^2 + b^2}$ and is therefore mod $(a + ib)$. The angle

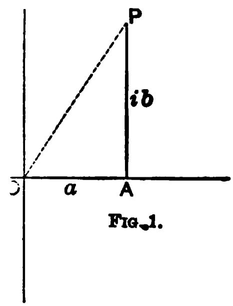


FIG. 1.

POA is such that $\cos POA = \frac{a}{\sqrt{a^2+b^2}}$, $\sin POA = \frac{b}{\sqrt{a^2+b^2}}$,

$\tan POA = \frac{b}{a}$; it is therefore the angle α in the expression

$$a + ib = m(\cos \alpha + i \sin \alpha).$$

[Given the representations of two complex quantities, represent their products and quotients. State in words how to do it.]

31. De Moivre's Formula.—In the equation

$$(x + iy)(a + ib) = mre^{i(\theta + \alpha)},$$

we may let $a + ib = x + iy$ and thus find that

$$(x + iy)^2 = r^2e^{i \cdot 2\theta} = r^2(\cos 2\theta + i \sin 2\theta);$$

similarly,

$$(x + iy)^n = r^n e^{i \cdot n\vartheta} = \cos n\vartheta + i \sin n\vartheta,$$

so long as n is a positive whole number. Under this restriction, then,

$$(\cos \vartheta + i \sin \vartheta)^n = \cos n\vartheta + i \sin n\vartheta,$$

a formula of very great importance, called De Moivre's formula.

32. Generalization of $e^{i \times \vartheta}$.—We know also that

$$\frac{x + iy}{a + ib} = \frac{r}{m} e^{i(\vartheta - \alpha)}.$$

If in this we make

$$(a + ib) = (x + iy)^{n+1} = r^{n+1} e^{i(n+1)\vartheta},$$

we shall have

$$\frac{x + iy}{(x + iy)^{n+1}} = (x + iy)^{-n} = \frac{r}{r^{n+1}} e^{i(\vartheta - (n+1)\vartheta)} = r^{-n} e^{-in\vartheta}.$$

Now, when $n = 1$, this gives

$$\frac{x + iy}{(x + iy)^2} = \frac{1}{(x + iy)} = \frac{1}{r} e^{-i\vartheta}.$$

But we know that

$$\frac{1}{x + iy} = \frac{x - iy}{x^2 + y^2} = \frac{r(\cos \vartheta - i \sin \vartheta)}{r^2} = \frac{1}{r} (\cos \vartheta - i \sin \vartheta).$$

We conclude that

$$e^{-i\vartheta} = \cos\vartheta - i\sin\vartheta,$$

whatever real angle ϑ may be. Hence

$$r^{-n}e^{-in\vartheta} = r^{-n}\cos n\vartheta - i\sin n\vartheta;$$

and finally, since

$$(x+iy)^{-n} = r^{-n}(\cos\vartheta + i\sin\vartheta)^{-n},$$

we have

$$(\cos\vartheta + i\sin\vartheta)^{-n} = \cos n\vartheta - i\sin n\vartheta.$$

We may now conclude that $(e^{i\vartheta})^n = e^{in\vartheta}$, where n is any positive or negative whole number. Suppose that $\vartheta = \frac{\phi}{n}$, then $e^{i\vartheta} = e^{i\frac{\phi}{n}}$ and $(e^{i\vartheta})^n = \left(e^{i\frac{\phi}{n}}\right)^n = e^{in\vartheta} = e^{i\phi}$. That is, the nth power of $e^{i\frac{\phi}{n}}$ is $e^{i\phi}$; conversely, one of the nth roots of $e^{i\phi}$ must be $e^{i\frac{\phi}{n}}$; hence $(\cos\vartheta + i\sin\vartheta)^{\frac{1}{n}} = \cos\frac{\vartheta}{n} + i\sin\frac{\vartheta}{n}$; but it has also other values, as we shall see.

Finally, if m and n are whole numbers, we have

$$(e^{i\vartheta})^{\frac{m}{n}} = e^{i\frac{m}{n}\vartheta} = \cos\frac{m}{n}\vartheta + i\sin\frac{m}{n}\vartheta;$$

but by choosing m and n properly we may make $\frac{m}{n}$ come as near as we please to any rational or irrational number what-

ever; hence, when x is any number whatever, integer, fraction, or irrational, we have

$$(e^{i\vartheta})^x = e^{ix\vartheta} = \cos x\vartheta + i \sin x\vartheta.$$

33. The Values of $(e^{i\vartheta})^{\frac{1}{n}}$. By definition we have

$$e^{2i\pi} = \cos 2\pi + i \sin 2\pi = 1;$$

hence

$$e^{i\vartheta} \cdot e^{2i\pi} = e^{i\vartheta} = e^{i(\vartheta + 2\pi)}.$$

But more generally still we have $e^{i\vartheta} = e^{i(\vartheta + 2k\pi)}$, where k is any whole number whatever. Hence we must have

$$(e^{i\vartheta})^{\frac{1}{n}} = \left(e^{i(\vartheta + 2k\pi)}\right)^{\frac{1}{n}} = e^{i\frac{\vartheta + 2k\pi}{n}};$$

whence

$$(e^{i\vartheta})^{\frac{1}{n}} = \cos \frac{\vartheta + 2k\pi}{n} + i \sin \frac{\vartheta + 2k\pi}{n},$$

where k may be any whole number. Apparently, then, the number of values of $(e^{i\vartheta})^{\frac{1}{n}}$ is infinite, but that is not so; for, making $k = n$, we obtain

$$(e^{i\vartheta})^{\frac{1}{n}} = \cos \left(\frac{\vartheta}{n} + 2\pi\right) + i \sin \left(\frac{\vartheta}{n} + 2\pi\right) = \cos \frac{\vartheta}{n} + i \sin \frac{\vartheta}{n},$$

which is no different from the result when $k = 0$. We conclude that when k has run through the numbers 0, 1, 2, . . . , $n - 1$, the values of $(e^{i\vartheta})^{\frac{1}{n}}$ begin to repeat themselves. Hence there are only n of them.

[Show this for $k = n + 1, n + 2, \ldots, n + h$.]

34. Remark.—It is useful to note that

$$e^{i\pi} = \cos\pi + i\sin\pi = -1;$$

$$e^{-i\pi} = \cos\pi - i\sin\pi = -1;$$

$$e^{i\frac{\pi}{2}} = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = i;$$

$$e^{-i\frac{\pi}{2}} = \cos\frac{\pi}{2} - i\sin\frac{\pi}{2} = -i.$$

35. The Equation $x^n = 1$.—Suppose we have given the equation

$$x^n = 1,$$

and are required to find its roots. We have

$$x^n = e^{2i\pi} = e^{i(2\pi + 2k\pi)}.$$

Hence

$$x = e^{i\frac{2\pi + 2k\pi}{n}} = \cos\frac{2\pi + 2k\pi}{n} + i\sin\frac{2\pi + 2k\pi}{n}.$$

The last value which k takes is $n - 1$; for $k = n - 1$, we have

$$x = \cos\frac{2n\pi}{n} + i\sin\frac{2n\pi}{n} = 1.$$

If n is even, we may make $k = \frac{n}{2} - 1$; then we have

$$x = \cos\frac{n\pi}{n} + i\sin\frac{n\pi}{n} = \cos\pi = -1.$$

Hence if n is even, both -1 and $+1$ are roots of $x^n = 1$. But if n is odd, $+1$ is the only real root.

36. The Equation $x^n = -1$.—To find the roots of

$$x^n = -1.$$

Here we have

$$x^n = e^{i(\pi + 2k\pi)}, \quad \text{since} \quad e^{i\pi} = -1.$$

Hence

$$x = e^{i\frac{\pi + 2k\pi}{n}} = \cos\frac{\pi + 2k\pi}{n} + i \sin\frac{\pi + 2k\pi}{n}.$$

If n is even, the roots are all imaginary, since no even power of a real quantity can be negative; but if n is odd, we may make $k = \frac{n-1}{2}$; then we find

$$x = \cos\pi + i\sin\pi = -1.$$

We conclude that when n is odd, there is one and only one real root, -1.

[Compute the square, cube, fourth, and fifth roots of $+1$ and -1.]

37. The Equations $x^n = \pm a$.—The equations $x^n = \pm a$ may be written $x^n = a(\pm 1)$; hence $x = \sqrt[n]{a}(\pm 1)^{\frac{1}{n}}$. We see therefore that algebraic equations of the form $x^n = \pm a$ which are called binomial equations, can always be solved, and have always n roots and no more.

39. Addition of Complex Quantities.—Let us return to Argand's diagram. We can represent the sum of $x + iy$ and $a + ib$ by first representing $x + iy$, which gives the point P; then, using P for a new origin, plot $a + ib$, thus

finding a new point, Q. OQ is the modulus of the sum, and QOX is the argument. In this way we may add quantity

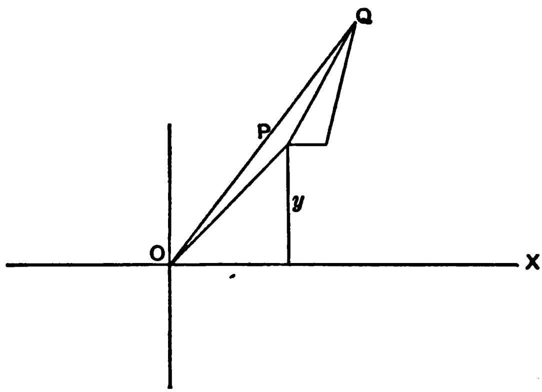


Fig. 2.

to quantity, and we see that the sum of any number of quantities will be represented by a point in the plane. Suppose we have thus added k quantities, and found a point

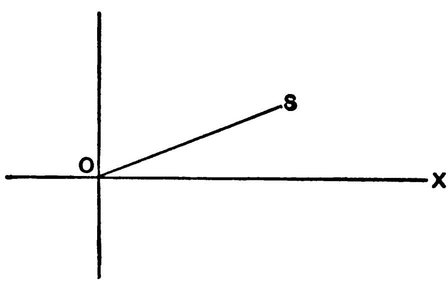


Fig. 3.

S representing their sum: OS is the modulus, and SOX the argument, of the sum. If the sum is zero, its modulus must be zero, and the point S must coincide with O. Hence the

moduli of the quantities added will form a closed polygon with one vertex at the origin. In the figure (Fig. 4) we

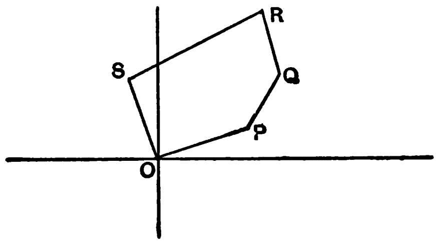


FIG. 4.

have added six quantities which have 0 for their sum. It can now be shown that:

40. Theorem.—Every algebraic equation has a root. Let

$$f(z) = a_0z^n + a_1z^{n-1} + a_2z^{n-2} + \ldots + a_{n-1}z + a_n = 0$$

be the equation. If it has a root, it must be real or complex or pure imaginary, hence it will be of the form $\alpha + i\beta$. Let $x + iy$ be any value given to z. Each term of $f(z)$ will

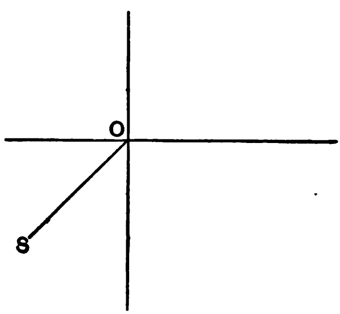


FIG. 5.

be a complex quantity, and we may find a point in the plane representing their sum, or $f(x + iy)$. If $x + iy$ is a root of $f(z)$, this point will be the origin, and the moduli of the

terms of $f(z)$ will form a closed polygon. If $x + iy$ is not a root, the value of $f(z)$ will be represented by some point S (Fig. 5), with OS for its modulus. Denote this modulus by r; what we are to show is that, for some value of $z = x + iy$, r will vanish.

Lemma.—We have

$$\text{mod}\, f(z) \overset{=}{<} \text{mod}\, a_0 z^n + \text{mod}\, a_1 z^{n-1} + \ldots + \text{mod}\, a_n.$$

Thence

$$\frac{\text{mod}\, f(z)}{\text{mod}\, a_0 z^n} = \text{mod}\, \frac{f(z)}{a_0 z^n} \overset{=}{<} \frac{\text{mod}\, a_0 z^n}{\text{mod}\, a_0 z^n} + \frac{\text{mod}\, a_1 z^{n-1}}{\text{mod}\, a_0 z^n}$$
$$+ \ldots + \frac{\text{mod}\, a_n}{\text{mod}\, a_0 z^n} = 1 + \text{mod}\, \frac{a_1 z^{n-1}}{a_0 z^n} + \text{mod}\, \frac{a_2 z^{n-2}}{a_0 z^n}$$
$$+ \ldots + \text{mod}\, \frac{a^n}{a_0 z^n}.$$

Now if we take mod $a_0 z^n$ large enough, all the terms after the first tend toward zero; so that for a proper value of z we shall have, as nearly as we please,

$$\text{mod}\, \frac{f(z)}{a_0 z^n} = 1, \quad \text{or} \quad \text{mod}\, f(z) = \text{mod}\, a_0 z^n;$$

that is,

$$r = \text{mod}\, a_0 z^n.$$

Hence by choosing mod z large enough we can make mod $f(z)$ as great as we please; in particular, we can make mod $f(z)$ exceed mod $f(w)$, when w has any value within a certain circle drawn about the origin as a centre; and if we make the circle large enough, we shall have always

$$\text{mod}\, f(z) > \text{mod}\, f(w),$$

where z is any point without the circle, and w any point within it.

Consider $f(w)$ and $f(w+\zeta)$, where mod ζ is very small. We have $f(w+\zeta)=f(w)+\zeta f'(w)+\zeta^2 R$, where R includes all the other terms of the development. Suppose that $f'(w)$ is not zero; then $f(w+\zeta)-f(w) = \zeta f'(w)+R\zeta^2$; the term $R\zeta^2$ being of the second order of infinitesimals. We may show the relation of our quantities by Argand's diagram.

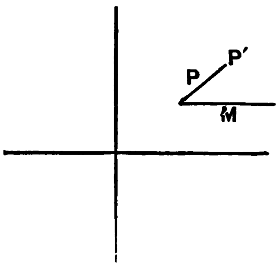


FIG. 6.

P represents $f(w)$; P', $f(w+\zeta)$; then $PP' =$ mod $\zeta f'(w)$, and the angle $P'PM =$ the argument of $\zeta f'(w)$. We neglect $R\zeta^2$ in this figure. Let $\zeta = \rho e^{i\phi}$ and $f'(w) = Ae^{i\alpha}$; then $\zeta f'(w) = A\rho e^{i(\alpha+\phi)}$. The angle $P'PM$ is therefore $\alpha+\phi$. Let us make ϕ vary, keeping ρ constant and w fixed, so that $A\rho = PP' =$

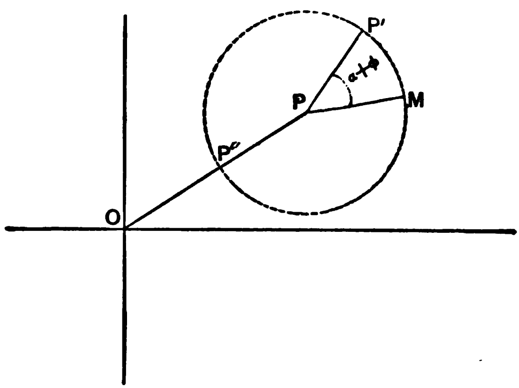


FIG. 7.

mod $\zeta f'(w)$ is of constant length. The point P' will describe a small circle of radius $A\rho$ about P. This circle will

intersect the line OP in a point P'' between O and P. Let the corresponding value of ϕ be ϕ_1, and of ζ be ζ_1; P'' represents $f(w+\zeta_1)$. Make $w+\zeta_1 = w_1$; then

$$f(w+\zeta_1) = f(w_1).$$

Since P'' lies between O and P, $OP'' < OP$; but

$$OP'' = \operatorname{mod} f(w_1); \quad \therefore \operatorname{mod} f(w_1) < \operatorname{mod} f(w).$$

Starting from P'', we may in the same way construct a point P''' representing $f(w_2)$, and lying between O and P'', so that $\operatorname{mod} f(w_2) < \operatorname{mod} f(w_1)$.

It was shown that if w is within and z without a very large circle whose centre is O, then $\operatorname{mod} f(w) < \operatorname{mod} f(z)$. Make $\operatorname{mod} f(z) \geqq M$, M being sufficiently large, then $\operatorname{mod} f(w) \leqq M$ for all points w. Hence $f(w)$ always lies within a circle of radius M, and much more is its least value within that circle; but so long as $\operatorname{mod} f(w) > 0$, we can always find a point w_1 such that $\operatorname{mod} f(w_1) < \operatorname{mod} f(w)$; hence the least value of $\operatorname{mod} f(w)$ is zero; hence $f(w)$ has a root.

41. Theorem.—We may now show that the equation $f(w) = 0$ has n roots. For let α_1 be the root which it must have by what precedes, then $\dfrac{f(w)}{w-\alpha} = Q_1$, where Q_1 is a polynomial of degree $n-1$. The equation $Q_1 = 0$ must also have a root, say α_2; hence $\dfrac{Q_1}{w-\alpha_2} = Q_2$, a polynomial of degree $n-2$. Hence

$$f(w) = Q_1(w-\alpha_1) = Q_2(w-\alpha_1)(w-\alpha_2).$$

It is clear from this that α_2 is a root of $f(w)$. Thus we may continue and show that

$$f(w) = (w-\alpha_1)(w-\alpha_2)\ldots(w-\alpha_n)a_0;$$

there being n linear factors of the form $w - \alpha_i$. The a_0 must come in in order to make the coefficient of w^n the same as in $f(w)$.

42. Relations between the Coefficients and Roots of $f(w) = 0$.—Multiplying together the three factors $(w - \alpha_1)$, $(w - \alpha_2)$, $(w - \alpha_3)$, we find for their product

$$w^3 - (\alpha_1 + \alpha_2 + \alpha_3)w^2 + (\alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_3\alpha_1)w - \alpha_1\alpha_2\alpha_3.$$

Hence in the product of three linear factors of the form $w - \alpha_i$,

1) The coefficient of w^3 is 1;
2) The coefficient of w^2 is the negative sum of the α's;
3) The coefficient of w is the sum of the products of the α's taken two and two;
4) The coefficient of w^0 is the product of the three α's.

We can generalize this result by multiplying in more factors, and shall find when the n factors have all been multiplied together that:

1) The coefficient of w^n is unity;
2) The coefficient of w^{n-1} is

$$-(\alpha_1 + \alpha_2 + \ldots + \alpha\ ) = -\Sigma\alpha;$$

3) The coefficient of w^{n-2} is

$$(\alpha_1\alpha_2 + \alpha_2\alpha_3 + \ldots) = +\Sigma\alpha_i\alpha_j;$$

4) The coefficient of w^{n-3} is $-\Sigma\alpha_i\alpha_j\alpha_k$;

$\cdot$

$\cdot$

$\cdot$

5) The coefficient of w^{n-l} is $(-1)^l\Sigma\alpha_i\alpha_j\ldots$ (l in a set);
6) The last term is $(-1)^n$ into the product of all the roots.

43. Restatement of Relations between Coefficients and Roots.—Now $(w - \alpha_1)(w - \alpha_2) \ldots (w - \alpha_n) = 0$ represents any equation which has unity for the coefficient of w^n; hence, in any such equation,

1) The coefficient of w^{n-1} is the negative sum of the roots;
2) The coefficient of w^{n-2} is the sum of the combinations of the roots two and two;
3) The coefficient of w^{n-3} is the negative sum of the combination of the roots three in a set;
4) The coefficient of w^{n-l} is $(-1)^l$ times the sum of the combination of the roots l in a set;

The coefficient of w^n may be made unity in

$$f(w) = a_0 w^n + a_1 w^{n-1} + \ldots = 0,$$

by dividing through by a_0. Let us write

$$\frac{f(w)}{a_0} = w^n + p_1 w^{n-1} + p_2 w^{n-2} + \ldots + p_n = 0,$$

where $p_1 = \dfrac{a_1}{a_0}$, $p_2 = \dfrac{a_2}{a_0}$, $\ldots$, $p_n = \dfrac{a_n}{a_0}$; then we shall have

$$\left.\begin{array}{r}
-p_1 = \Sigma\alpha; \\
p_2 = \Sigma\alpha_i\alpha_j; \\
-p_3 = \Sigma\alpha_i\alpha_j\alpha_k; \\
\cdot \\
\cdot \\
\cdot \\
(-1)^l p_l = \Sigma\alpha_i\alpha_j \ldots (l \text{ in a set}).
\end{array}\right\} \quad \ldots \quad (A)$$

44. Example.—In the cubic $x^3 - 2x^2 - 3x + 1 = 0$, if α_1, α_2, α_3 be the roots, we have

$$2 = \alpha_1 + \alpha_2 + \alpha_3;$$

$$-3 = \alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_3\alpha_1;$$

$$-1 = \alpha_1\alpha_2\alpha_3.$$

[Express coefficients of several equations in terms of the (unknown) roots.]

45. To form an Equation whose Roots are given.—Equations (A) enable us to form an equation when its roots are given. For instance, suppose the roots of a cubic are 2, $3 + 5i$, $3 - 5i$: we shall have

$$-p_1 = 2 + 3 + 5i + 3 - 5i = 8;$$

$$p_2 = 2(3 + 5i) + 2(3 - 5i) + (3 + 5i)(3 - 5i) = 46;$$

$$-p_3 = 2(3 + 5i)(3 - 5i) = 68.$$

Hence the equation is

$$w^3 - 8w^2 + 46w - 68 = 0.$$

[Form equations whose roots are given.]

46. To lower the Degree of an Equation.—When one or more roots of an equation are known, it may be divided by the corresponding linear factors; the quotient equated to zero forms the equation from which to find the remaining roots. Thus

$$x^4 - 6x^3 + 8x^2 - 17x + 10 = 0$$

has 5 for a root. Dividing by $x-5$, we see that the other three roots may be found by solving

$$x^3 - x^2 + 3x - 2 = 0.$$

We find the coefficients of the quotient as before shown:

1	-6	8	-17	10
	5	-5	15	-10
	-1	3	-2	0

[$x^4 + 4x^3 + 6x^2 + 4x + 5 = 0$ has i for a root; reduce it to a cubic. $x^6 - x^5 - 8x^4 + 2x^3 + 21x^2 - 9x - 54 = 0$ has $\sqrt{2} + i$ for a root; reduce it to a biquadratic by one division.]

47. Definition.—A symmetric function of the roots of an equation is one whose value is not changed by any permutations among the roots.

If the roots of a cubic are $\alpha_1, \alpha_2, \alpha_3$, then $\alpha_1 + \alpha_2 + \alpha_3$ is a symmetric function. It is not altered, for instance, if we permute α_1 and α_3.

All the coefficients of an equation are symmetric functions of the roots.

Symmetric functions are denoted by writing Σ before a specimen term:

$$\Sigma\alpha^2 = \alpha_1^2 + \alpha_2^2 + \alpha_3^2;$$

$$\Sigma\alpha_i^2\alpha_j = \alpha_1^2\alpha_2 + \alpha_1^2\alpha_3 + \alpha_2^2\alpha_3 + \alpha_2^2\alpha_1 + \alpha_3^2\alpha_1 + \alpha_3^2\alpha_2.$$

[What do $\Sigma\alpha_1\alpha_2$, $\Sigma\alpha_1\alpha_2\alpha_3$, $\Sigma\alpha_1^2\alpha_2$, denote for the biquadratic?]

48. Remark.—Rational symmetric functions of the roots of an equation can always be expressed in terms of the

coefficients. For example, if the cubic be $x^3 + p_1x^2 + p_2x + p_3 = 0$ and the roots $\alpha_1, \alpha_2, \alpha_3$, let us express $\Sigma\dfrac{\alpha_1^2 + \alpha_2^2}{\alpha_1\alpha_2}$ in terms of the coefficients. We have

$$\Sigma\frac{\alpha_1^2 + \alpha_2^2}{\alpha_1\alpha_2^2} = \Sigma\frac{p_1^2 - 2p_2 - \alpha_3^2}{-\dfrac{p_3}{\alpha_3}} = \Sigma\frac{\alpha_3(p_1^2 - 2p_2)}{-p_3} + \Sigma\frac{\alpha_3^3}{p_3}$$

$$= \frac{p_1}{p_3}(p_1^2 - 2p_2) + \Sigma\frac{\alpha_1^3}{p_3}$$

$$= \frac{p_1}{p_3}(p_1^2 - 2p_2) + \frac{-p_1^3 + 3p_1p_2 - 3p_3}{p_3} = \frac{p_1p_2 - 3p_3}{p_3}.$$

[Compute for the same cubic $\Sigma(\beta + \gamma - \alpha)^3$.]

49. Problem.—To express the sums of the same powers of the roots in terms of the coefficients.

$$\text{We have}\quad f(x) = (x - \alpha_1)(x - \alpha_2)(x - \alpha_3)\ldots(x - \alpha_n)$$
$$= x^n + p_1x^{n-1} + p_2x^{n-2} + \ldots + p_n.$$

From the first form of $f(x)$ we have

$$\begin{aligned} f'(x) &= (x - \alpha_2)(x - \alpha_3)\ldots(x - \alpha_n) \\ &\quad + (x - \alpha_1)(x - \alpha_3)(x - \alpha_4)\ldots(x - \alpha_n) + \ldots \\ &\quad + (x - \alpha_1)(x - \alpha_2)\ldots(x - \alpha_{n-1}) \\ &= \frac{f(x)}{x - \alpha_1} + \frac{f(x)}{(x - \alpha_2)} + \ldots + \frac{f(x)}{x - \alpha_n}; \end{aligned}$$

also,

$$f'(x) = nx^{n-1} + (n-1)p_1x^{n-2} + (n-2)p_2x^{n-3} + \ldots + p_{n-1},$$

from the second form of $f(x)$.

Now $\dfrac{f(x)}{x-\alpha}$, where α is any root, gives a quotient

$$x^{n-1} + q_1x^{n-2} + q_2x^{n-3} + \dots + q_{n-1},$$

where we find the q's as follows:

$$\begin{array}{lllll}
1 & p_1 & p_2 & p_3 \dots & p_n \\
 & \alpha & \alpha(p_1+\alpha) & \alpha(p_2+\alpha p_1+\alpha^2) & \\
\hline
 & p_1+\alpha, & p_2+\alpha p_1+\alpha^2, & p_3+\alpha p_2+\alpha^2 p_1+\alpha^3, \dots &
\end{array}$$

Hence

$$\frac{f(x)}{x-\alpha} = \begin{array}{r|r|r|r}
x^{n-1}+\alpha & x^{n-2}+\alpha^2 & x^{n-3}+\ \alpha^3 & x^{n-4} \dots +\alpha^{n-1} \\
+p_1 & +p_1\alpha & +p_1\alpha^2 & +p_1\alpha^{n-2} \\
 & +p_2 & +p_2\alpha & +p_2\alpha^{n-3} \\
 & & +p_3 & +\dots \\
 & & & \dots \\
 & & & +p_{n-2}\alpha \\
 & & & +p_{n-1}
\end{array}$$

In this expression replace α by $\alpha_1, \alpha_2, \dots, \alpha_n$ successively and add the results together. The left member becomes $\Sigma \dfrac{f(x)}{x-\alpha} = f'(x)$. In the right member we may for brevity write $\Sigma\alpha = s_1$, $\Sigma\alpha^2 = s_2, \dots, \Sigma\alpha^i = s_i$, and we get

$$\begin{aligned}
nx^{n-1} &+ (s_1 + np_1)x^{n-2} + (s_2+p_1s_1+np_2)x^{n-3} \\
&+ (s_3 + s_2p_1 + s_1p_2 + np_3)x^{n-4} + \dots \\
&+ (s_{n-1} + p_1s_{n-2} + \dots + p_{n-2}s_1 + np_{n-1}.
\end{aligned}$$

Comparing this expression with the second form of $f'(x)$, we find

$$\begin{aligned}
(n-1)p_1 &= s_1 + np_1;\\
(n-2)p_2 &= s_2 + p_1s_1 + np_2;\\
(n-3)p_3 &= s_3 + p_1s_2 + p_2s_1 + np_3;\\
&\;\;\vdots\\
p_{n-1} &= s_{n-1} + p_1s_{n-2} + \dots + p_{n-2}s_1 + np_{n-1};
\end{aligned}$$

whence

$$\begin{aligned}
&s_1 + p_1 = 0;\\
&s_2 + s_1p_1 + 2p_2 = 0;\\
&s_3 + s_2p_1 + s_1p_2 + 3p_3 = 0;\\
&\qquad\vdots\\
&s_{n-1} + p_1s_{n-2} + \dots + s_1p_{n-2} + (n-1)p_{n-1} = 0.
\end{aligned}$$

There are $n-1$ of these equations from which to find the $n-1$ quantities $s_1, s_2, \dots, s_{n-1}$. From the first,

$$s_1 = -p_1.$$

From the first two by determinants,

$$s_2 = \begin{vmatrix} -p_1 & 1 \\ -2p_2 & p_1 \end{vmatrix} \div \begin{vmatrix} 0 & 1 \\ 1 & p_1 \end{vmatrix} = p_1^2 - 2p_2.$$

From the first three by determinants,

$$s_3 = \begin{vmatrix} -p_1 & 0 & 1 \\ -2p_2 & 1 & p_1 \\ -3p_3 & p_1 & p_2 \end{vmatrix} \div \begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & p_1 \\ 1 & p_1 & p_2 \end{vmatrix} = \begin{vmatrix} p_1 & 0 & 1 \\ 2p_2 & 1 & p_1 \\ 3p_3 & p_1 & p_2 \end{vmatrix}.$$

And in short,

$$s_i = \pm \begin{vmatrix} p_1 & 0 & 0 & & 1 \\ 2p_2 & 0 & 0 & & p_1 \\ 3p_3 & 0 & 0 & & p_2 \\ 4p_4 & 0 & 0 & & p_3 \\ \cdot & & 1 & & \cdot \\ \vdots & 1 & p_1 & \cdot & \cdot \\ ip_i & p_1 & p_2 & & p_{i-1} \end{vmatrix},$$

the divisor being always 1 in absolute value. This law of formation holds good for all values of i up to $n-1$. To find s_n, s_{n+1}, . . . , we have

$$x^{m-n}f(x) = x^m + p_1x^{m-1} + p_2x^{m-2} + \ldots + p_nx^{m-n}.$$

In this replace x by α_1, α_2, . . . , α_n, in succession and add the results. All the left members vanish; hence we have

$$s_m + p_1s_{m-1} + p_2s_{m-2} + \ldots + p_ns_{m-n} = 0.$$

Giving to m the values n, $n+1$, $n+2$, . . . , we find

$$s_n + p_1s_{n-1} + \ldots + p_ns_1 = 0, \text{ etc.}$$

Thus we can find s_i, where i is any positive integer.

50. Problem.—To form an equation whose roots shall be the reciprocals of those of

$$f(x) = x^n + p_1x^{n-1} + p_2x^{n-2} + \ldots + p_{n-1}x + p_n = 0.$$

If we make $x = \dfrac{1}{y}$, then if α be a value of x, the corresponding value of y is $\dfrac{1}{\alpha}$. Hence the equation satisfied by

y has roots which are the reciprocals of those of $f(x)$. Make $x = \frac{1}{y}$ in $f(x)$ and clear of fractions; the result is

$$p_n y^n + p_{n-1} y^{n-1} + p_{n-2} y^{n-2} + \ldots + p_1 y + 1 = 0,$$

or, after dividing by p_n,

$$\phi(y) = y^n + \frac{p_{n-1}}{p_n} y^{n-1} + \ldots + \frac{p_1}{p_n} y + \frac{1}{p_n} = 0.$$

51. Sum of the ith Powers of the Reciprocals of the Roots of $f(x) = 0$.—We may express the sum of the ith powers of the roots of this equation in terms of its coefficients as before; but this sum will be the sum of the ith powers of the reciprocals of the roots of $f(x)$; and since the coefficients of $\phi(y)$ are functions of those of $f(x)$, we shall have the sum of the ith powers of the reciprocals of the roots of $f(x)$ expressed in terms of the coefficients of $f(x)$.

For $\phi(y)$, $$s_3 = -\begin{vmatrix} \frac{p_{n-1}}{p_n} & 0 & 1 \\ \frac{2p_{n-2}}{p_n} & 1 & \frac{p_{n-1}}{p_n} \\ \frac{3p_{n-3}}{p_n} & \frac{p_{n-1}}{p_n} & \frac{p_{n-2}}{p_n} \end{vmatrix} = \Sigma\frac{1}{\alpha^3}$$

if $\alpha_1, \alpha_2, \ldots, \alpha_n$, are the roots of $f(x)$.

[Form the sum of squares, cubes, fourth powers of the roots of various equations and their reciprocals.]

52. Symmetric Functions can always be expressed rationally in Terms of the Coefficients.—A fractional rational symmetric function of the roots is the quotient of two integral ones; hence if an integral rational symmetric

function of the roots can always be expressed rationally in terms of the coefficients, any other rational symmetric function of the roots can be so expressed.

Now any integral rational symmetric function is necessarily either a simple symmetric expression in the roots, or a sum of such expressions. Hence its most general form can be no other than

$$A\Sigma\alpha_1{}^p\alpha_2{}^q + B\Sigma\alpha_1{}^{p'}\alpha_2{}^{q'}\alpha_3{}^{r'} + C\Sigma\alpha_1{}^{p''}\alpha_2{}^{q''}\alpha_3{}^{r''}\alpha_4{}^{s''} + \ldots,$$

where A, B,. . . are numerical coefficients.

If each Σ can be expressed rationally in terms of the coefficients of $f(x)$, the whole function can be so expressed. But

$$\Sigma\alpha_1{}^p\alpha_2{}^q = \Sigma\alpha^p \,.\, \Sigma\alpha^q - \Sigma\alpha^{p+q} = s_ps_q - s_{p+q},$$

$$\begin{aligned}\Sigma\alpha_1{}^{p'}\alpha_2{}^{q'}\alpha_3{}^{r'} &= \Sigma\alpha_1{}^{p'}\alpha_2{}^{q'}.\Sigma\alpha_3{}^{r'} - \Sigma\alpha_1{}^{p'+r'}\alpha_2{}^{q'} - \Sigma\alpha_1{}^{p'}\alpha_2{}^{q'+r'}\\ &= (s_{p'}s_{q'} - s_{p'+q'})s_{r'} - (s_{p'+r'}s_{q'} - s_{p'+q'+r'})\\ &\qquad - (s_{p'}s_{q'+r'} - s_{p'+q'+r'})\\ &= s_{p'}s_{q'}s_{r'} - s_{p'+q'}s_{r'} - s_{p'+r'}s_{q'} - s_{q'+r'}s_{p'}\\ &\qquad + 2s_{p'+q'+r'}.\end{aligned}$$

These equations, if they are not evident, may be verified by actual multiplication; and in a similar way we may express any Σ in terms of s's; but, since the s's are all rational in the coefficients, the same is true of the Σ's and therefore of the whole integral symmetric function.

[For the cubic $x^3 + p_1x^2 + p_2x + p_3 = 0$ express in terms of the coefficients the symmetric functions $\Sigma(\alpha_1{}^2 - \alpha_2{}^2)^2$; $\Sigma\dfrac{\alpha_1{}^2 + \alpha_2{}^2}{\alpha_1 + \alpha_2}$; $\Sigma\dfrac{\alpha_1{}^2 + \alpha_2\alpha_3}{\alpha_2 + \alpha_3}$; $\Sigma\left(\dfrac{\alpha_1 - \alpha_2}{\alpha_1 + \alpha_2}\right)^2$.]

53. Definition.—A simple symmetric function of the roots is always homogeneous in the roots, and its *weight* is its degree in the roots. When expressed in terms of the

coefficients it is not generally homogeneous; the degree of the term of highest degree in the coefficients is called the order of the function.

54. Theorem.—The order of a symmetric function is the same as the highest power of any root contained in it.

To show this we recall that $p_1 = \frac{a_1}{a_0}, \ldots, p_n = \frac{a_2}{a_0}$. Now any function of $p_1, \ldots, p_n$, not already homogeneous, will become so if we replace p_1 by $\frac{a_1}{a_0}, \ldots, p_n$ by $\frac{a_n}{a_0}$, and then multiply every term by the highest power of a_0 which occurs in any denominator. Suppose this highest power is a^ω; ω will clearly be the degree of ϕ in $p_1, \ldots, p_n$. Then any rational and integral function of $p_1, \ldots, p_n$ will satisfy the equation

$$\phi(p_1, \ldots, p_n) = \frac{1}{a_0^\omega}\Phi(a_0, a_1, \ldots, a_n),$$

where Φ is what ϕ becomes when we replace p_1 by $\frac{a_1}{a_0}, \ldots,$ and then multiply every term by a_0^ω. Φ is therefore homogeneous in the coefficients, and ω is its degree.

But we know that rational and integral symmetric functions of the roots can always be expressed as rational and integral functions of the coefficients. Let $\phi(p_1, \ldots, p_n)$, then, be the expression in the coefficients of a symmetric function of the roots $\alpha_1, \ldots, \alpha_n$; that is,

$$\phi(p_1, \ldots, p_n) = \Psi(\alpha_1, \ldots, \alpha_n).$$

Then we have

$$a_0^\omega \Psi(\alpha_1, \ldots, \alpha_n) = \Phi(a_0, a_1, \ldots, a_n).$$

The coefficients of the equation whose roots are $\frac{1}{\alpha_1}, \ldots,$ $\frac{1}{\alpha_n}$, are $a_n, a_{n-1}, \ldots, a_0$; hence for this equation

$$a_n^{\omega}\Psi\left(\frac{1}{\alpha_1}, \ldots, \frac{1}{\alpha_n}\right) = \Phi(a_n, a_{n-1}, \ldots, a_0).$$

But

$$\Psi\left(\frac{1}{\alpha_1}, \ldots, \frac{1}{\alpha_n}\right) = \frac{\theta(\alpha_1, \ldots, \alpha_n)}{(\alpha_1\alpha_2 \ldots \alpha_n)^r}.$$

We get this form by reducing all the terms of Ψ to a common denominator and taking the denominator out. But

$$(\alpha_1 . \alpha_2 . \ldots . \alpha_n)^r = \left(\frac{a_n}{a_0}\right)^r;$$

hence

$$\Psi\left(\frac{1}{\alpha_1}, \ldots, \frac{1}{\alpha_n}\right) = \left(\frac{a_0}{a_n}\right)^r \theta(\alpha_1, \ldots, \alpha_n).$$

Hence

$$a_n^{\omega}\left(\frac{a_0}{a_n}\right)^r \theta(\alpha_1, \ldots, \alpha_n) = \Phi(a_n, \ldots, a_0).$$

The left member contains $a_n^{\omega - r}$ as a factor; that is, it is divisable by a power of a_n unless $\omega = r$; but if we consider how Φ was constructed we see that Φ is not divisible by any power of a_n; hence a_n cannot be a factor in the left member. We conclude that $\omega = r$.

Now r is the highest power to which $\frac{1}{\alpha_1}, \ldots, \frac{1}{\alpha_n}$, occur in $\Psi\left(\frac{1}{\alpha_1}, \ldots, \frac{1}{\alpha_n}\right)$; hence r is the highest degree to which $\alpha_1, \ldots, \alpha_n$, occur in $\Psi(\alpha_1, \ldots, \alpha_n)$. Moreover ω is the degree of $\phi(p_1, \ldots, p_n)$ in $p, \ldots, p_n$; and finally

$\phi(p_1, \ldots, p_n) = \Psi(\alpha_1, \ldots, \alpha_n)$. We conclude, since $\omega = r$, that the highest power of any one root in a symmetric function is its degree when expressed in the coefficients.

55. Theorem.—When a simple symmetric function is expressed in terms of the coefficients, the weight of every term is the same and is equal to the degree in the roots of the symmetrical function. To show this, let $\Sigma\alpha_1^p\alpha_2^q \ldots \alpha_i^l = f(p_1, p_2, \ldots, p_n)$. If we multiply each root by λ, the left member is multiplied by $\lambda^{p+q+\cdots+l}$; hence every term of the right member must be multiplied by $\lambda^{p+q+\cdots+l}$; hence every term is of degree $p+q+\ldots+l$ in the roots; that is, every term is of weight $p+q+\ldots+l$.

56. Elimination. The Resultant.—Let

$$f(x) = 0 = x^n + p_1x^{n-1} + \ldots + p_n$$

and

$$\phi(x) = 0 = x^m + q_1x^{m-1} + \ldots + q_m$$

be two algebraic equations. It is required to find a function of the coefficients which shall vanish if $f(x) = 0$ and $\phi(x) = 0$ have a common root, but not otherwise. This function is called the *resultant* of $f(x)$ and $\phi(x)$.

57. Computation of the Resultant by Symmetric Functions.—Let the roots of $f(x) = 0$ be $\alpha_1, \alpha_2, \ldots, \alpha_n$; and the roots of $\phi(x) = 0$ be $\beta_1, \beta_2, \ldots, \beta_m$. If there is a common root, let it be β_i; then β_i must equal some α, say α_j; and we must have $f(\alpha_j) = f(\beta_i) = 0$. Of course in general we do not know whether or not there is a common root or what root it is; but if we form the product $f(\beta_1)$, $f(\beta_2), \ldots, f(\beta_m)$, some one of the factors must vanish if there is a common root, and therefore the whole product will vanish. Now $f(\beta_1), f(\beta_2), \ldots, f(\beta_m)$ is a rational and integral symmetric function of $\beta_1, \beta_2, \ldots, \beta_m$. [Why?]

We can therefore express it rationally in terms of the coefficients of $\phi(x)$. It already contains the coefficients of $f(x)$; hence the result will contain the coefficients of $\phi(x)$ and $f(x)$, and, since it will vanish if they have a common root, it is the resultant sought.

58. Example.—For an example, let

$$f(x) = x^3 - 6x^2 + 11x - 6 = 0,$$

and

$$\phi(x) = x^2 - 6x + 5 = 0.$$

To form their resultant, let the roots of $\phi(x)$ be β_1, β_2; the resultant, which may be called R, is

$$\begin{aligned} R &= (\beta_1^3 - 6\beta_1^2 + 11\beta_1 + 6)(\beta_2^3 - 6\beta_2^2 + 11\beta_2 - 6) \\ &= \beta_1^3\beta_2^3 - 6\beta_1^2\beta_2^2(\beta_1 + \beta_2) + 11\beta_1\beta_2(\beta_1^2 + \beta_2^2) \\ &\qquad - 6(\beta_1^3 + \beta_2^3) + 36\beta_1^2\beta_2^2 - 66\beta_1\beta_2(\beta_1 + \beta_2) \\ &\qquad + 36(\beta_1^2 + \beta_2^2) + 121\beta_1\beta_2 - 66(\beta_1 + \beta_2) + 36. \end{aligned}$$

Now $\beta_1\beta_2 = 5$; $\beta_1 + \beta_2 = 6$; $\beta_1^2 + \beta_2^2 = 26$; $\beta_1^3 + \beta_2^3 = 126$. Hence

$$\begin{aligned} R &= 125 - 6.25.6 + 11.5.26 - 6.126 + 36.25 - 66.5.6 \\ &\quad + 36.26 + 121.5 - 66.6 + 36 = 0. \end{aligned}$$

We conclude that $f(x) = 0$ and $\phi(x) = 0$ have a common root.

[Test several pairs of equations for common roots.]

59. Remark.—We could just as well find the resultant by substituting the roots of $f(x)$ in $\phi(x)$; the two results would differ only by a numerical factor, which is of no consequence.

60. Remark.—R contains the coefficients of $f(x)$ to the degree m, and those of $\phi(x)$ to the degree n.

61. Resultant by Sylvester's Dialytic Method.—In the above work we have eliminated x from the two equations

$f(x) = 0$, $\phi(x) = 0$ by symmetric functions. We may also eliminate by Sylvester's dialytic method, as follows:

$$f(x) = a_0 x^n + a_1 x^{n-1} + \ldots + a = 0,$$
$$\phi(x) = b_0 x^m + b_1 x^{m-1} + \ldots + b_m = 0.$$

Multiply $f(x) = 0$ by $x^{m-1}, x^{m-2}, \ldots x, x^0$, successively; we thus form m equations in $x^{m+n-1}, x^{m+n-2}, \ldots, x^0$. Now multiply $\phi(x) = 0$ by $x^{n-1}, x^{n-2}, \ldots x, x^0$, successively; we thus form n equations in $x^{m+n-1}, x^{m+n-2}, \ldots, x^0$. Regarding each power of x as an unknown quantity, we have now $m + n$ equations in $m + n$ unknown quantities. If they are consistent, their determinant must vanish; but they will not be consistent unless $\phi(x)$ and $f(x)$ have a common root. Hence this determinant is another form of the resultant.

The form of this determinant is:

$$\left|\begin{array}{llllllll}
a_0 & a_1 & & \ldots a_n & 0 & & \ldots & 0 \\
0 & a_0 & & \ldots a_{n-1} & a_n & & \ldots & 0 \\
0 & 0 & a_0 & \ldots a_{n-2} & a_{n-1} & a_n & \ldots & 0 \\
\multicolumn{8}{l}{\text{(The coefficients of } f(x) \text{ forming } m} \\
\multicolumn{8}{l}{\text{rows, each row beginning one place}} \\
\multicolumn{8}{l}{\text{farther to the right than the preced-}} \\
\multicolumn{8}{l}{\text{ing one.)}} \\
b_0 & b_1 & & \ldots b_m & & & \ldots & 0 \\
0 & b_0 & b_1 & \ldots b_{m-1} & b_m & & \ldots & 0 \\
\multicolumn{8}{l}{\text{(The coefficients of } \phi(x) \text{ forming } n} \\
\multicolumn{8}{l}{\text{rows; each row beginning one place}} \\
\multicolumn{8}{l}{\text{farther to the right than the preced-}} \\
\multicolumn{8}{l}{\text{ing one.)}}
\end{array}\right| = R.$$

62. Remark.—There are $m + n$ rows and $m + n$ columns.

63. Example.—By this method the resultant of

$$f(x) = x^3 - 6x^2 + 11x - 6 = 0$$

and

$$\phi(x) = x^2 - 6x + 5 \qquad = 0$$

is

$$R = \begin{vmatrix} 1 & -6 & 11 & -6 & 0 \\ 0 & 1 & -6 & 11 & -6 \\ 1 & -6 & 5 & 0 & 0 \\ 0 & 1 & -6 & 5 & 0 \\ 0 & 0 & 1 & -6 & 5 \end{vmatrix} = 0.$$

[Eliminate x from several systems of two equations.]

64. Double Roots.—If an equation $f(x) = 0$ has a double root, say α_1, it will contain the factor $(x - \alpha_1)^2$, and the first derivative of $f(x)$ will contain the factor $(x - \alpha_1)$. Hence if $f(x) = 0$ has a double root, then $f(x) = 0$ and $f'(x) = 0$ have that root for a common root. To find, then, whether or not an equation has a double root, we form the resultant of $f(x) = 0$ and $f'(x) = 0$. If the resultant vanishes, $f(x)$ has a double root, but not otherwise.

65. Example.—Let $f(x) = x^3 - 7x^2 + 16x - 12 = 0$; then

$$f'(x) = 3x^2 - 14x + 16.$$

Their resultant is

$$\begin{vmatrix} 1 & -7 & 16 & -12 & 0 \\ 0 & 1 & -7 & 16 & -12 \\ 3 & -14 & 16 & 0 & 0 \\ 0 & 3 & -14 & 16 & 0 \\ 0 & 0 & 3 & -14 & 16 \end{vmatrix},$$

which $= 0$. Hence $f(x) = 0$ has a double root.

66. The Discriminant.—Suppose we form the resultant of $f(x) = 0$ and $f'(x) = 0$ when

$$f(x) = a_0x^3 + a_1x^2 + a_2x + a_3 = 0;$$

it will be

$$\begin{vmatrix} a_0 & a_1 & a_2 & a_3 & 0 \\ 0 & a_0 & a_1 & a_2 & a_3 \\ 3a_0 & 2a_1 & a_2 & 0 & 0 \\ 0 & 3a_0 & 2a_1 & a_2 & 0 \\ 0 & 0 & 3a_0 & 2a_1 & a_2 \end{vmatrix}.$$

We see that this resultant contains a_0 as a factor; in fact, the resultant of $f(x) = 0$ and $f'(x) = 0$ always contains the factor a_0. The quotient obtained by dividing out a_0 is called the *discriminant* of $f(x)$. Omitting unimportant factors we can put the discriminant in the form

$$\begin{vmatrix} a_0 & a_1 & a_2 & a_3 \\ a_1 & 2a_2 & 3a_3 & 0 \\ 3a_0 & 2a_1 & a_2 & 0 \\ 0 & 3a_0 & 2a_1 & a_2 \end{vmatrix}$$

and easily see by counting that it is in general of degree $2(n - 1)$ in the coefficients of $f(x)$.

[Form the discriminants of several functions.]

67. To find the Common Root.—Knowing that $f(x) = 0$ and $\phi(x) = 0$ have a common root, we may find it as follows:

Denoting by R the resultant of

$$f(x) = a_0x^n + a_1x^{n-1} + \ldots + a_n = 0$$

and

$$\phi(x) = b_0x^m + \ldots + b^m = 0,$$

we shall have, by giving simultaneous increments to a_k and a_l,

$$R(a_k + da_k,\ a_l + da_l) = R + \frac{\delta R}{\delta a_k} da_n + \frac{\delta R}{\delta a_l} da_l$$

by Taylor's Theorem. If da_k and da_l be so chosen that

$$da_k\alpha^{n-k} + da_l\alpha^{n-l} = 0,$$

where α is the common root of $f(x) = 0$ and $\phi(x) = 0$, α will still be a common root after the change in a_k and a_l, and the new resultant must therefore vanish; that is,

$$R + \frac{\delta R}{\delta a_k} da_k + \frac{\delta R}{\delta a_l} da_l = 0.$$

But $R = 0$ and $da_k : da_l :: -\alpha^{n-l} : \alpha^{n-k}$. Hence

$$\frac{\delta R}{\delta a_k} : \frac{\delta R}{\delta a_l} :: \alpha^{n-k} : \alpha^{n-l}.$$

If now $l = k + 1$, this becomes

$$\frac{\delta R}{\delta a_k} : \frac{\delta R}{\delta a_{k+1}} :: \alpha^{n-k} : \alpha^{n-k-1} :: \alpha : 1,$$

whence

$$\alpha = \frac{\delta R}{\delta a_k} \div \frac{\delta R}{\delta a_{k+1}},$$

where k is any suffix.

Hence the common root is equal to the ratio of the derivatives of the resultant with respect to any two consecutive coefficients.

The same rule applied to the discriminant of $f(x)$ gives the value of the repeated root in $f(x)$.

[Compute common and repeated roots.]

68. Caution.—If there is more than one common root, or if $f(x) = 0$ has a multiple root of multiplicity greater than two, or has more than one multiple root, these rules do not apply.

PART III.

COMPUTATION OF THE REAL ROOTS OF NUMERICAL EQUATIONS.

1. Descartes' Rule of Signs.—By the aid of this rule we can find a superior limit to the number of positive and negative real roots of an equation; if we know how many real roots an equation has, we can say how many of them are positive and how many are negative.

Suppose the factors corresponding to all the imaginary and all the negative roots of $f(x) = 0$ to have been multiplied together; in the product there will be a certain arrangement of signs, say $+ + - +$, for instance. Let α be a positive real root; the signs of $(x - \alpha)$ are $+ -$. Multiplying in this new factor and attending only to signs, we have

$$\begin{array}{ccccc} & + & + & - & + \\ & & & + & - \\ \hline & - & - & + & - \\ + & + & - & + & \\ \hline + & \pm & - & + & - \end{array}$$

One of the signs of the product is ambiguous, but whether it be $+$ or $-$ there is none the less a change from $+$ to $-$ among the first three signs. There are two more changes among the other signs of the product, making three in all. But before multiplying in $x - \alpha$ there were only two changes of sign, from $+$ to $-$ and $-$ to $+$. Hence the factor corresponding to the positive root α has increased

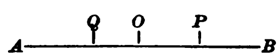
the number of changes of sign in the product by at least one. Each factor corresponding to a positive root will have the same effect; and we must therefore conclude that the number of changes of sign from $+$ to $-$ and from $-$ to $+$ in $f(x)$ will at least equal the number of positive roots of $f(x) = 0$; hence the number of positive roots of $f(x) = 0$ may equal but can never exceed the number of variations of sign in $f(x)$. In general, the number of positive roots is less than the number of variations of sign in $f(x)$.

The negative roots of $f(x) = 0$ are the positive roots of $f(-x) = 0$; we may therefore add that the number of negative roots of $f(x) = 0$ cannot exceed the number of variations of sign in $f(-x)$.

These two statements constitute Descartes' Rule of Signs.

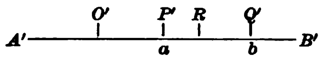
2. Theorem.—If a and b be real quantities and $f(a)$ is positive while $f(b)$ is negative, there must be at least one real root of $f(x) = 0$ between a and b.

If we represent positive real values by points on the right of O in the line AB, and negative values by points on the left, then O itself represents the value 0. $f(a)$ is rep-



resented by a point P on the right of O, and $f(b)$ by a point Q on the left of O. It is clear that $f(x)$ can never pass continuously from P to Q along AB without passing through O. The value of x corresponding to this point is a real root of $f(x) = 0$ and it lies between a and b.

The values of x may be represented by points on another line $A'O'B'$; the value a will be some point P', and b some other point Q'. The root value is somewhere between P' and Q', say at R. Then as x moves from P' to R on $A'B'$,



$f(x)$ moves from P to O on AB; and as x moves from R to Q', $f(x)$ moves from O to Q.

3. Rolle's Theorem.—Between two consecutive real roots of $f(x)=0$ there is always at least one real root of $f'(x)=0$.

Let α and β be consecutive real roots of $f(x)=0$; then choosing the sign of h so that $\alpha+h$ is nearer to β than α is, and taking h very small,

$$f(\alpha+h)=f(\alpha)+hf'(\alpha)=hf'(\alpha),$$
$$f(\beta-h)=f(\beta)-hf'(\beta)=-hf'(\beta).$$

But $f(\alpha+h)$ and $f(\beta-h)$ have the same sign, since there is no real root of $f(x)=0$ between α and β.

It follows that $f'(\alpha)$ and $f'(\beta)$ must have different signs; consequently there is at least one real root of $f'(x)=0$ between α and β.

4. Multiple Roots.—If $f(x)$ is divisible by $(x-\alpha)^k$, α is a multiple root of multiplicity k of $f(x)=0$. We can then write

$$f(x)=(x-\alpha)^k\psi(x),$$

and $\psi(x)=0$ will not have α for a root. Differentiating this equation, we find

$$f'(x)=k(x-\alpha)^{k-1}\psi(x)+(x-\alpha)^k\psi'(x).$$

Hence $f'(x)$ contains $(x-\alpha)^{k-1}$ as a factor; and therefore we conclude that a root of $f(x)=0$ of multiplicity k will be a root of $f'(x)=0$ of multiplicity $k-1$.

Let $\phi(x)$ be the greatest common divisor of $f(x)$ and $f'(x)$; all the roots common to $f(x)=0$ and $f'(x)=0$ will be multiple roots of $\phi(x)=0$ of multiplicity less by one

than in $f(x) = 0$; conversely, by the definition of the greatest common divisor, $\phi(x) = 0$ can have no other roots.

[Find the multiple roots of several equations.]

5. Theorem.—If α be a real root of $f(x) = 0$ but not of $f'(x) = 0$, and h a very small positive quantity, then $f'(\alpha - h)$ and $f(\alpha - h)$ have opposite signs, while $f'(\alpha + h)$ and $f(\alpha + h)$ have the same signs.

In fact,

$$f(\alpha - h) = f(\alpha) - hf'(\alpha) = -hf'(\alpha),$$

while $f(\alpha + h) = hf'(\alpha)$. Now $f'(\alpha + h)$ and $f'(\alpha - h)$ have both the same sign as $f'(\alpha)$. Hence just before x, varying continuously from less to greater values, reaches the root value, $f(x)$ and $f'(x)$ have opposite signs, and just afterwards $f(x)$ and $f'(x)$ have the same sign.

This theorem is true, as we shall show, even if α is a root of $f'(x) = 0$.

6. Definition.—A superior limit of the positive roots of an equation is a positive number which is greater than any of the roots.

There are several rules for finding this limit; sometimes one gives the closer result, sometimes the other. Of course the smaller the number found the better, so long as we are certain of its being a superior limit.

7. Rule for finding a Superior Limit to the Roots.—One rule for finding a superior limit is the following:

If $f(x) = 0 \equiv x^n + p_1x^{n-1} + \ldots$, and if the first negative term be $-p_rx^{n-r}$, and the greatest negative coefficient be $-p_k$, then a superior limit of the positive roots of $f(x) = 0$ is $\sqrt[r]{p_k} + 1$.

Note first that after x, varying continuously, becomes greater than the greatest positive root, $f(x)$ is always positive, since $x = +\infty$ makes $f(x)$ positive, and between ∞

and the greatest positive root $f(x)$ does not change sign. We conclude that any quantity such that all greater quantities make $f(x)$ positive is a superior limit of the positive roots. Now if x is so taken that

$$x^n > p_k(x^{n-r} + x^{n-r-1} + \ldots + x + 1) = p_k\frac{x^{n-r+1} - 1}{x - 1},$$

$f(x)$ will be always positive. But

$$x^n > p_k\frac{x^{n-r+1} - 1}{x - 1}, \quad \text{if} \quad x^n > p_k\frac{x^{n-r+1}}{x - 1},$$

or

$$x^{n+1} - x^n > p_k x^{n-r+1};$$

or, taking $x > 1$,

$$x - 1 > p_k x^{-r+1},$$

or

$$x^{r+1} - x^r > p_k x,$$

or

$$x^r - x^{r-1} > p_k;$$

that is,

$$x^{r-1}(x - 1) > p_k.$$

But

$$x^{r-1}(x - 1) > p_k, \quad \text{if} \quad (x - 1)^{r-1}(x - 1) > p_k,$$

since $x^{r-1} > (x - 1)^{r-1}$.

Hence $f(x)$ will be always positive if we choose x so as to satisfy the inequality

$$x - 1 > \sqrt[r]{p_k}, \quad \text{or} \quad x > \sqrt[r]{p_k} + 1.$$

We conclude that $\sqrt[r]{p_k} + 1$ is a superior limit of the positive roots.

[Find a superior limit to the positive roots of several equations by this method.]

9. Remarks.—Practically the superior limit is found without much difficulty by trial; the object is to determine the smallest whole number such that all greater numbers make $f(x)$ positive.

[Find the superior limit by trial in a variety of cases.]

It lessens the number of trials required to compute first $\sqrt[r]{p_k}+1$; of course no greater number need be considered, but a smaller one may be a closer limit. It is very often, though by no means always, true that the whole number next less than the greatest positive root will make $f(x)$ negative; in this case the closest limit is found by adding 1 to the greatest whole number which makes $f(x)$ negative. Still, if two real roots should lie between two consecutive whole numbers, they would give no sign of themselves, and this test might lead one astray. Sturm's Theorem enables us to say just how many real roots lie between any two numbers whatever.

10. Sturm's Theorem.—Suppose first that $f(x)=0$ and $f'(x)=0$ have no common roots. Proceed as if finding the greatest common divisor of $f(x)$ and $f'(x)$, only change the sign of each remainder before using it as a divisor. Let $q_1, q_2, \ldots q_{n-1}$ be the successive quotients; $f_2(x)$, $f_3(x)$, $\ldots, f_n(x)$, the remainders with their signs changed; then

$$
\begin{aligned}
f(x) &= q_1 f'(x) - f_2(x),\\
f'(x) &= q_2 f_2(x) - f_3(x),\\
f_2(x) &= q_3 f_3(x) - f_4(x),\\
&\cdot \qquad \cdot \qquad \cdot\\
&\cdot \qquad \cdot \qquad \cdot\\
&\cdot \qquad \cdot \qquad \cdot\\
f_{n-2}(x) &= q_{n-1} f_{n-1}(x) - f_n(x),
\end{aligned}
$$

since each dividend is equal to the product of the divisor and quotient plus the remainder. $f_n(x)$ is a number and

free from x. The quantities $f(x)$, $f'(x)$, $f_2(x)$, ... , $f_n(x)$ are known as the Sturmian Functions.

By hypothesis $f(x)$ and $f'(x)$ have no common factor; in other words, they never vanish for the same value of x. In consequence of this, we see from the table that no two consecutive Sturmian functions can vanish together. It is also evident from the table that if one of them vanishes for a certain value of x, say α, the preceding and following ones have opposite signs for $x = \alpha$. Now let $f_r(x)$ be the function which vanishes for α, that is $f_r(\alpha) = 0$. Then $f_{r-1}(\alpha) = -f_{r+1}(\alpha)$. Moreover neither $f_{r-1}(x) = 0$ nor $f_{r+1}(x) = 0$ has a root infinitesimally near to α; hence they have the same signs when x is a little less than α as they have when x is a little greater than α, because a function can only change sign by passing through the point zero. We conclude that just before x, varying continuously, reaches the value α, the signs of $f_{r-1}(x)$, $f_r(x)$, $f_{r+1}(x)$ must be $+ \pm -$ or $- \pm +$, and just afterwards they must be $+ \mp -$ or $- \mp +$; that is, $f_r(x)$ changes sign as x passes α and the other two do not. The point to notice is that the number of changes from $+$ to $-$ and from $-$ to $+$ in this series of signs is not altered by the vanishing of $f_r(x)$. And generally; if the signs of $f(x)$, $f'(x)$, $f_2(x)$, ... , $f_n(x)$ be written down in a row, and if the changes from $+$ to $-$ and from $-$ to $+$ in this row be counted, the number so found is not altered by any changes of sign which the vanishing of $f'(x)$, $f_2(x)$, ... , $f_n(x)$ can introduce into the row. The signs will change about in the row, but the number of variations from $+$ to $-$ and from $-$ to $+$ will be a constant. But as often as $f(x)$ passes through zero a variation of sign is lost from the row; for we already know that just before x, varying continuously from less to greater values, passes a root of $f(x)$, the signs of $f(x)$ and $f'(x)$ are opposite, and just after x

passes the root of $f(x)$, the signs of $f(x)$ and $f'(x)$ are the same.

We conclude that if a number b be substituted in the Sturmian functions, and the corresponding number of variations of sign in the row described above be h; and if another number a, $a > b$, be substituted giving g variations,—then the whole number of real roots of $f(x) = 0$ lying between b and a is equal to $h - g$. If the number of variations given by $-\infty$ is p, by 0 is q, and by $+\infty$ is r, then the whole number of real roots of $f(x) = 0$ is $p - r$, the number of positive roots is $q - r$, and the number of negative roots is $p - q$. This is Sturm's Theorem.

Observe that in substituting $+\infty$ or $-\infty$, the sign of the term of highest degree is the sign of the function.

11. Separation of the Real Roots.—To separate the roots of an equation is to find pairs of numbers such that each pair includes one and only one root. To do this, begin with 0 and substitute in the Sturmian functions 0, 1, 2, . . . in succession until the same row of signs is found as that given by $+\infty$. If one variation is lost in passing from any number to the next one, a root lies between them; and the integral part of that root is the smaller of the two numbers. For example, if a variation is lost in passing from 8 to 9, there is a root between 8 and 9, and its first figure is 8.

If two variations were lost in passing say from 8 to 9, then it would be necessary to substitute 8.1, 8.2, and so on, until the two roots lying between 8 and 9 were separated.

To separate the negative roots of $f(x) = 0$, we begin with 0 and substitute in succession $0, -1, -2, \ldots,$ in the Sturmian functions. A root is passed whenever the number of variations is increased by unity.

12. Extension of Sturm's Theorem.—If $f(x) = 0$ and

$f'(x) = 0$ have a greatest common divisor, it will be one of the Sturmian functions, say $f_r(x)$, and the other functions, $f_{r+1}(x), \ldots,$ will not exist. We shall have

$$\begin{aligned} f(x) &= q_1 f'(x) + f_2(x), \\ f'(x) &= q_2 f_2(x) - f_3(x), \\ f_2(x) &= q_3 f_3(x) - f_4(x), \\ &\cdot \\ &\cdot \\ &\cdot \\ f_{r-2}(x) &= q_{r-1} f_{r-1}(x) - f_r(x). \end{aligned}$$

Lemma.—Let α be a multiple root of multiplicity k, of $f(x) = 0$; then α is a root of $f'(x) = 0$ and of $f_r(x) = 0$ of multiplicity $k - 1$. We have

$$f(x) = (x - \alpha)^k \Psi(x)$$

and

$$f'(x) = (x-\alpha)^{k-1}[(x-\alpha)\Psi'(x) + k\Psi(x)] = (x-\alpha)^{k-1}\theta(x);$$

where neither $\Psi(x)$ nor $\theta(x)$ has α for a root. Now, if h is small,

$$f(\alpha + h) = f(\alpha) + hf'(\alpha) + \frac{h^2}{2!}f''(\alpha) + \ldots + \frac{h^k}{k!}f^k(\alpha);$$

where $f(\alpha), f'(\alpha), \ldots, f^{k-1}(\alpha)$ are all zero but $f^k(\alpha)$ does not vanish. The remainder of the series is negligible. Also

$$f'(\alpha + h) = f'(\alpha) + hf''(\alpha) + \ldots + \frac{h^{k-1}}{\overline{k-1}!}f^k(\alpha).$$

That is, since $f(\alpha) = f'(\alpha) = \dots = f^{k-1}(\alpha) = 0$,

$$f(\alpha + h) = \frac{h^k}{k!} f^k(\alpha) \quad \text{and} \quad f'(\alpha + h) = \frac{h^{k-1}}{\underline{k-1}!} f^k(\alpha).$$

From this we conclude that if h is negative (that is, just before $x = \alpha$), $f(\alpha + h)$ and $f'(\alpha + h)$ have opposite signs; but if h is positive, $f(\alpha + h)$ and $f'(\alpha + h)$ have the same sign. It follows that when x passes a multiple root of any multiplicity whatever, there is a variation of sign lost in the Sturmian functions.

It is important to note that $f'(x)$, $f_2(x)$, ..., $f_r(x)$ all contain the factor $(x-\alpha)^{k-1}$. The expansions by Taylor's Series of $f'(x+h)$, $f_2(x+h)$, ..., $f_r(x+h)$, where h is small, must therefore end with $\frac{h^{k-1}}{\underline{k-1}!} f^k(\alpha)$, ..., $\frac{h^{k-1}}{\underline{k-1}!} f_2{}^{k-1}(\alpha)$; hence they all change sign, or else none of them changes sign, as h passes from $-h$ to $+h$, and we conclude that as x passes the multiple root α no variation is lost or gained by the vanishing of $f'(x)$, ..., $f_r(x)$.

Resuming we may say:

If in forming the Sturmian functions there is no remainder after $f_r(x)$, then $f(x) = 0$ has multiple roots which may be found by solving $f_r(x) = 0$.

The number of variations of sign lost in substituting b and a successively, $a > b$, is the number of real roots lying between b and a; but each multiple root counts only once.

13. Examples.—In forming the Sturmian functions positive numerical factors may be introduced or removed without affecting the result, just as in finding the greatest common divisor.

For an application of the Sturmian functions let it be required to find the number and situation of the real roots

of $f(x) = x^4 - 2x^3 - 3x^2 + 10x - 4 = 0$. We have within numerical factors:

$$f'(x) = 2x^3 - 3x^2 - 3x + 5,$$
$$f_2(x) = 9x^2 - 27x + 11,$$
$$f_3(x) = -8x - 3,$$
$$f_4(x) = -1433.$$

The last function being a number, there are no multiple roots. Substituting $-\infty$, 0, $+\infty$ in $f(x), f'(x)$, we obtain the rows of signs—

$-\infty$	$+$	$-$	$+$	$+$	$-$	3 variations.
0	$-$	$+$	$+$	$-$	$-$	2 variations.
$+\infty$	$+$	$+$	$+$	$-$	$-$	1 variation.

Hence there are two real roots, one negative and one positive. We can find where these roots lie by substituting whole numbers in $f(x)$ alone, since there is no danger of two roots lying between the same consecutive integers. $f(x)$ changes sign in passing from 0 to 1, and in passing from -2 to -3; hence the positive root lies between 0 and 1, and the negative root between -2 and -3.

Again, let
$$f(x) = x^3 - 7x + 7,$$
$$f'(x) = 3x^2 - 7,$$
$$f_2(x) = 2x - 3,$$
$$f_3(x) = 1.$$

We obtain for the rows of signs:

$-\infty$	$-$	$+$	$-$	$+$	Three variations,
0	$+$	$-$	$-$	$+$	Two variations,
$+\infty$	$+$	$+$	$+$	$+$	No variations.

There are three real roots, one negative and two positive. To separate the roots we have:

	$f(x)$	$f'(x)$	$f_2(x)$	$f_3(x)$	
-4	$-$	$+$	$-$	$+$	
-3	$+$	$+$	$-$	$+$	One variation lost.
1	$+$	$-$	$-$	$+$	
2	$+$	$+$	$+$	$+$	Two variations lost.

The negative root lies between -4 and -3; and both positive roots lie between 1 and 2.

To separate the positive roots we have:

1.1	$+$	$-$	$-$	$+$	
1.2	$+$	$-$	$-$	$+$	
1.3	$+$	$-$	$-$	$+$	
1.4	$-$	$-$	$-$	$+$	One variation lost.

Hence the smaller positive root is $1.3+$. By a similar process we can find the second figure of the other.

To investigate the roots of

$$f(x) \equiv x^6 - 7x^5 + 15x^4 - 40x^3 + 48x - 16 = 0.$$

Here $f'(x) = 6x^5 - 35x^4 + 60x^3 - 80x + 48$,

$f_2(x) = 13x^4 - 84x^3 + 192x^2 - 176x + 48$,

$f_3(x) = (x-2)^3$.

It turns out that there are no other remainders; hence $(x-2)^3$ is the greatest common divisor of $f(x)$ and $f'(x)$, and 2 is a multiple root of $f(x) = 0$ of multiplicity four,

We have the following table of signs:

$-\infty$	$+$	$-$	$+$	$-$	
0	$-$	$+$	$+$	$-$	One variation lost, $\therefore$ one negative root.
$+\infty$	$+$	$+$	$+$	$+$	Two variations lost, $\therefore$ two positive roots.

Thus there are two different positive roots, but one of them is repeated so as to count for four roots; the whole number of real roots is six.

[Separate the roots of at least six equations.]

14. Conditions for all the Roots Real.—The conditions that $-\infty$ may give n variations of sign in the Sturmian functions are that none of the functions be lacking, and that each have its first term positive; under these conditions $+\infty$ will give no variations, and we shall conclude that all the roots of the equation are real.

15. Remark.—By Sturm's theorem we can find the first one or two figures of each real root of $f(x)=0$, and might go on to find as many more as were desired if the work of substituting did not increase so rapidly with each new figure. The process of approximating to the incommensurable roots has been greatly simplified by methods now to be described.

16. Theorem.—In the first place, if the coefficients of $f(x)$ be all whole numbers, and the coefficient of x^n be unity no root of $f(x)=0$ can be a rational fraction.

For, let

$$f(x) = x^n + px^{n-1} + \ldots = 0,$$

where $p_1, p_2, \ldots$ are all whole numbers; and if possible let $\frac{a}{b}$, a rational fraction in its lowest terms, be a root of $f(x)=0$. Then a and b are integers. Put $\frac{a}{b}$ for x in $f(x)=x$; multiply the result by b^{n-1}, and transpose the first term to the right member. We have

$$p_1a^{n-1} + p_2a^{n-2}b + \ldots + p_nb^{n-1} = -\frac{a^n}{b}.$$

Now the right member is a fraction, since a and b have no common divisor, while the terms of the left member are all integers. Hence our result is absurd, and no rational fraction can be a root of $f(x) = 0$.

17. Theorem. Generalization of the last Article.—Every equation can be put in the form

$$x^n + p_1 x^{n-1} + \ldots = 0,$$

where $p_1, p_2, \ldots$ are all integers and the coefficient of x^n is 1. Suppose the equation to stand

$$f(x) = x^n + q_1 x^{n-1} + \ldots = 0,$$

where q_1, $q_2, \ldots$ are fractions whose least common denominator is m; make $x = \dfrac{y}{m}$; substitute this value in the equation, and multiply through by m^n. The result is

$$y^n + mq_1 y^{n-1} + m^2 q_2 y^{n-2} + \ldots + m^n q_n = 0,$$

where the coefficients are all integers and that of y^n is 1.

[Apply this to several equations with fractional coefficients.]

18. Commensurable Roots.—The equation being in the form $x^n + p_1 x^{n-1} + \ldots + p_n = 0$, where $p_1, \ldots, p_n$ are all integers, the roots are either irrational, or else they are whole numbers. If whole numbers, they are among the divisors of p_n. Therefore, to find the commensurable roots, we substitute successively the positive and negative divisors of p_n in $f(x) = 0$, using only those which lie within the limits given by Sturm's theorem for the roots.

19. Example.—Suppose, for example, we wish to find the commensurable roots of $x^4 + x^3 - 2x^2 + 4x - 24 = 0$, knowing that they must lie between -4 and 3. The only admissible numbers are -3, -2, -1, $+1$, $+2$. We try -3 by the known method of substituting:

$$\begin{array}{rrrrrl} 1 & 1 & -2 & 4 & -24 & \\ & -3 & +6 & -12 & +24 & \\ \hline & -2 & +4 & -8 & 0 & = \text{rem.} \end{array}$$

Hence -3 is a root. Try now -2:

$$\begin{array}{rrrrrl} 1 & 1 & -2 & 4 & -24 & \\ & -2 & +2 & 0 & -8 & \\ \hline & -1 & 0 & 4 & 16 & = \text{rem.} \end{array}$$

Hence -2 is not a root. Try $+2$:

$$\begin{array}{rrrrrl} 1 & 1 & -2 & 4 & -24 & \\ & 2 & 6 & 8 & 24 & \\ \hline & 3 & 4 & 12 & 0 & = \text{rem.} \end{array}$$

Hence $+2$ is a root.

We can now divide the equation by $(x+3)(x-2)$ and thus reduce it to a quadratic.

[Test not less than six equations for commensurable roots.]

20. Horner's Method.—In approximating to the irrational roots of an equation by Horner's method, the following process is employed: Determine the first two or three figures of a root of $f(x) = 0$ in any manner; then form an equation whose roots are less than those of $f(x)$ by the figures found. From this equation the next figure can be easily found. Form the equation whose roots are less

by the last figure; from this find a new figure, and repeat the process as often as is desirable.

21. To Diminish the Roots.—To diminish the roots of $f(x) = 0$ by a number h we have only to make $x = y + h$ and substitute in $f(x)$. We have

$$f(x) = f(y+h) = f(y) + hf'(y) + \frac{h^2}{2!}f''(y) + \ldots + h^n.$$

The roots of the equation $f(y + h) = 0$ are less by h than those of $f(x) = 0$.

22. Computation of the New Coefficients.—If we arrange $f(y + h)$ with respect to y instead of h, it takes the form $f(x) = f(y + h) = A_0y^n + A_1y^{n-1} + \ldots + A_n$. Now since $y = x - h$, this can be written

$$f(x) = A_0(x-h)^n + A_1(x-h)^{n-1} + \ldots + A_{n-1}(x-h) + A_n,$$

identically. Hence $A_n = f(h)$; and the quotient after dividing $f(x)$ by $x - h$ is

$$A_0(x-h)^{n-1} + A_1(x-h)^{n-2} + \ldots + A_{n-1}.$$

A_{n-1} is the remainder after dividing a second time oy $x - h$; in fact, $A_n, A_{n-1}, A_{n-2}, \ldots A_1$ are the remainders left after dividing $f(x)$ by $x - h$ once, twice, ..., n times, successively; while $A_0 = a_0$ if $f(x) = a_0x_n + \ldots$. Now we know how to make these divisions rapidly:

$$\begin{array}{cccccc}
a_0 & a_1 & a_2 & \ldots & \ldots & a_n \\
 & ha_0 & hq_1 & & \ldots & hq_{n-1} \\
\hline
a_0 & q_1 & q_2 & q_{n-1} & & A_n \\
 & ha_0 & hr_1 & hr_{n-2} & & \\
\hline
a_0 & r_1 & r_2 & A_{n-1}, & &
\end{array}$$

where $a_0, a_1, \ldots, a_n$ are the coefficients of $f(x)$;

$a_0, q_1, q_2, \ldots, q_{n-1}$ are the coefficients of the first quotient and A_n is the first remainder;

$a_0, r_1, r_2, \ldots, r_{n-2}$ are the coefficients of the second quotient and A_{n-1} is the second remainder; and so on until all the remainders are found.

The coefficients of the equation $A_0 y^n + A_1 y^{n-1} + \ldots = 0$ are then completely known.

23. To find a New Figure of the Root.—We can thus rapidly diminish the roots of an equation by a given quantity. The next step is to find a new figure of the root. Suppose h is the part of the root found by trial, and that $h + y$ is the true value of the root; then if y is small,

$$0 = f(h + y) = f(h) + yf'(h),$$

to quantities of higher order. Hence, approximately,

$$y = -\frac{f(h)}{f'(h)}.$$

Comparing the two expansions of $f(h + y)$,

$$f(h) + yf'(h) + \ldots + a_0 y^n$$

and

$$A_n + A_{n-1}y + \ldots + A_0 y^n,$$

we see that, as hinted above, $f(h) = A_n$, $f'(h) = A_{n-1}$; hence

$$y = -\frac{f(h)}{f'(h)} = -\frac{A_n}{A_{n-1}}$$

—a value of y true generally to not more than one figure, that is, to the next figure of the root. We then proceed to

diminish the roots of the equation $A_0 y^n + \ldots = 0$ by this figure. A new equation,

$$B_0 z_n + B_1 z^{n-1} + \ldots + B_{n-1} z + B_n = 0,$$

is obtained, and the next figure of the root is the first figure of the quotient, $-\dfrac{B_n}{B_{n-1}}$. If this correction should be negative, as sometimes happens, the figure given by $-\dfrac{A_n}{A_{n-1}}$ is probably not right and must be corrected. Fortunately B_n and B_{n-1} are the two coefficients first computed; if their signs are the same we must diminish the last figure of the root. If $-\dfrac{B_n}{B_{n-1}}$ gives a figure in the same decimal place with the first figure of $-\dfrac{A_n}{A_{n-1}}$, careful attention must be paid to the fact to avoid mistakes. Usually 9 will then be the next figure of the root. It is not safe to rely on the correction $-\dfrac{A_n}{A_{n-1}}$; as a general thing a better one can be found by trial.

24. Examples and Remarks.—Let it be required to find the positive root of

$$f(x) = 4x^3 - 13x^2 - 31x - 275 = 0.$$

We know by Descartes' Rule of Signs that there can be only one positive root, and we find by trial that it lies between 6 and 7. Hence we take 6 for h and proceed to form the equation

$$A_0 y^n + A_1 y^{n-1} + \ldots,$$

whose roots are less by 6 than those of $f(x) = 0$. This is the process:

$$
\begin{array}{llll}
4 = A_0 & -13 & -31 & -275 \quad \underline{|6\quad} \\
 & 24 & 66 & 210 \\
 & \overline{11} = q_1 & \overline{35} = q_2 & \overline{-65} = A_3 \\
 & 24 & 210 & \\
 & \overline{35} = r_1 & \overline{245} = A_2 & \\
 & 24 & & \\
 & \overline{59} = A_1 & &
\end{array}
$$

$= 0$ had a root between 6 and 7; hence

$$4y_3 + 59y^2 + 245y - 65 = 0,$$

whose roots are less by 6, must have a root between 0 and 1. By trial it is seen to lie between .2 and .3. Hence the next figure of our root is .2; so far as found it is 6.2. The first figure of $-\dfrac{A_n}{A_{n-1}} = \dfrac{65}{245}$ is also .2. We now form the equation $B_0z^n + B_1z^{n-1} + \ldots = 0$.

$$
\begin{array}{llll}
4 = B_0 & 59 & 245 & -.65 \quad \underline{|6.2} \\
 & \underline{.8} & \underline{11.96} & 51.392 \\
 & 59.8 & 256.96 & -\overline{13.608} = B_3 \\
 & \underline{.8} & \underline{12.12} & \\
 & 60.6 & 269.08 = B_2 & \\
 & \underline{.8} & & \\
 & 61.4 = B_1 & &
\end{array}
$$

Observe that we here diminish the roots by .2, not by 6.2. To find the next figure we have $-\dfrac{B_3}{B_2} = \dfrac{13.608}{269.08} = .05$.

We shall now diminish the roots of $4z^3 + 61.4z^2 + 269.08z - 13.608 = 0$ by .05.

4	61.4	269.08	− 13.608	6.25
	.2	3.08	13.608	
	61.6	272.16	0.000	

It turns out that .05 is a root of $4z^3 + 61.4z^2 + 269.08z - 13.608 = 0$, hence we need go no further; 6.25 is the exact root of $f(x) = 0$. It would have been better to keep this whole process together, thus:

4	−13	−31	− 275	6.25
	24	66	210	
	11	35	− 65*	
	24	210	51.392	
	35	245*	− 13.608*	
	24	11.96	13.608	
	59*	256.96	0.000	
	.8	12.12		
	59.8	269.08*		
	.8	3.08		
	60.6	272.16		
	.8			
	61.4*			
	.2			
	61.6			

The coefficients of the successive new equations are marked with stars; there is no need of writing them down in a separate place. The next to the last coefficient in each one is the *trial divisor* by means of which the next figure of the root is found.

A root of $x^4 + 4x^3 - 4x^2 - 11x + 4 = 0$ lies between 1 and 2; compute it to four decimal places. It will be noticed that here $-\dfrac{A_n}{A_{n-1}}$ is very far from giving the second

figure, .6; in fact we find by trial that $y^4+8y^3+14y^2-3y-6=0$ has a root between .6 and .7; hence .6 is the second figure of the root of $f(x)=0$. The other figures are found accurately by division. The successive new coefficients are marked with stars.

1	4	−4	−11	4	1.6369
	1	5	1	−10	
	5	1	−10	−6*	
	1	6	7	5.0976	
	6	7	−3*	−.9024*	
	1	7	11.496	.72690561	
	7	14*	8.496	−.17549439*	
	1	5.16	14.808	.152131052016	
	8*	19.16	23.304*	−.023363337984*	
	.6	5.52	.926187		
	8.6	24.68	24.230187		
	.6	5.88	.935601		
	9.2	30.56*	25.165788		
	.6	.3129	.189387336		
	9.8	30.8729	25.355175336		
	.6	.3138	.189766488		
	10.4*	31.1867	25.544941824*		
	.03	.3147			
	10.43	31.5014*			
	.03	.063156			
	10.46	31.564556			
	.03	.063192			
	10.49	31.627748			
	.03	.063228			
	10.52*	31.690976*			
	.006				
	10.526				
	.006				
	10.532				
	.006				
	10.538				
	.006				
	10.544*				

There is a root of $f(x) = 0$ lying between 0 and 1; when we diminished all the roots of $f(x) = 0$ by 1 this root became negative; hence the absolute term -6, which is the product of the new roots, has the opposite sign to 10, the product of the original roots. Unless for a reason such as this, the absolute term cannot change its sign; if it does change, it is generally because a mistake has been made.

[Compute the roots of several equations to four places.]

25. Negative Roots.—If a negative root of $f(x) = 0$ is desired, let its value be $-\alpha$. Now the roots of $f(-x) = 0$ have opposite signs to those of $f(x) = 0$; hence α is a positive root of $f(-x) = 0$. Hence to compute the negative roots of $f(x) = 0$, we form $f(-x) = 0$ and compute its positive roots by the method already given. Clearly to form $f(-x)$ we have only to change the signs of those terms in $f(x)$ whose exponents are odd numbers.

26. By the method given above any real root of a numerical equation can be computed to any desired degree of accuracy. The solution of equations with literal coefficients is an entirely different problem and has not yet been accomplished for any but the cubic and biquadratic. The problem is, to find a function of the coefficients, involving only the symbols of algebra, which shall express indifferently any one of the n roots of $f(x) = 0$ by virtue of the ambiguity of the radical signs involved. It has been shown that this cannot be done.

The solutions of the cubic and biquadratic may be found in all good algebras, and it is therefore considered hardly necessary to reproduce them here.

NOTE ON RECTANGULAR ARRAYS.

An array of quantities having more columns than rows like the following:

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{vmatrix} = D,$$

is called a matrix. We shall consider only those having one more column than row. If such a matrix contains n columns, we can, by leaving out one column at a time, form from it n determinants of order $n-1$. Let

$$\begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 & \delta_1 \\ \alpha_2 & \beta_2 & \gamma_2 & \delta_2 \\ \alpha_3 & \beta_3 & \gamma_3 & \delta_3 \end{vmatrix} = D_1$$

be another matrix of four columns and three rows. We shall define the product DD_1 to mean

$$\begin{vmatrix} a_1\alpha_1+b_1\beta_1+c_1\gamma_1+d_1\delta_1, \ldots, a_1\alpha_3+b_1\beta_3+c_1\gamma_3+d_1\delta_3 \\ \vdots \\ a_3\alpha_1+b_3\beta_1+c_3\gamma_1+d_3\delta_1, \ldots, a_3\alpha_3+b_3\beta_3+c_3\gamma_3+d_3\delta_3 \end{vmatrix} = P,$$

which is a determinant of the third order formed by multiplying the rows of D into the rows of D_1 and taking the sums of the products for its constituents. We can break

up P into a sum of four determinants, of which a specimen one may be put in the form

$$\begin{vmatrix} a_1\alpha_1+b_1\beta_1+c_1\gamma_1, & a_1\alpha_2+b_1\beta_2+c_1\gamma_2, & a_1\alpha_3+b_1\beta_3+c_1\gamma_3 \\ a_2\alpha_1+b_2\beta_1+c_2\gamma_1, & a_2\alpha_2+b_2\beta_2+c_2\gamma_2, & a_2\alpha_3+b_2\beta_3+c_2\gamma_3 \\ a_3\alpha_1+b_3\beta_1+c_3\gamma_1, & a_3\alpha_2+b_3\beta_2+c_3\gamma_2, & a_3\alpha_3+b_3\beta_3+c_3\gamma_3 \end{vmatrix}.$$

We thus see that $DD_1 = P$ is the sum of the products of each determinant of D by the *corresponding* determinant of D_1.

EXAMPLES.*

PART I.

Art. 16.

I.
$$\begin{vmatrix} 9 & 13 & 17 & 4 \\ 18 & 28 & 33 & 8 \\ 30 & 40 & 54 & 13 \\ 24 & 37 & 46 & 11 \end{vmatrix} = -15.$$

II.
$$\begin{vmatrix} 5 & -10 & 11 & 0 \\ -10 & -11 & 12 & 4 \\ 11 & 12 & -11 & 2 \\ 0 & 4 & 2 & -6 \end{vmatrix} = 8100.$$

III.
$$\begin{vmatrix} 7 & -2 & 0 & 5 \\ -2 & 6 & -2 & -2 \\ 0 & -2 & 5 & 3 \\ 5 & 2 & 3 & 4 \end{vmatrix} = -972.$$

IV.
$$\begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix} = 2abc.$$

Art. 22.—If A_1, B_1, C_1, etc., are the first minors of

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \Delta, \text{ show that } \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \times \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} = \Delta^3.$$

* These examples are inserted merely as suggestions. It is hoped that instructors will add a great many more in several instances in their class work.

If Δ be any determinant of order n, and Δ_1 the determinant of its first minors, show that $\Delta\Delta_1 = \Delta^n$.

Art. 25.—Solve the following equations by determinants:

I. $$\begin{cases} x + 2y + 3z = 6 \\ 2x + 4y + z = 7 \\ 3x + 2y + 9z = 14 \end{cases}$$ *Ans.* $x = y = z = 1$.

II. $$\begin{cases} 4x + 7y + 3z - 3w = 9 \\ 2x - y - 4z + 3w = 13 \\ 3x + 2y - 7z - 4w = 2 \\ 5x - 3y + z + 5w = 13 \end{cases}$$

Art. 27.—Determine λ so that the following equations may be consistent:

I. $$\begin{cases} 6x - 7\lambda y = 0 \\ 11\lambda x + y = 0 \end{cases}$$

II. $$\begin{cases} 4x + 5y + z = 0 \\ 3\lambda x - 2y - z = 0 \\ 6x - \lambda y - 13z = 0 \end{cases}$$

Art. 29.—Eliminate x from the equations:

I. $$\begin{cases} x^3 + x^2y + xy^2 + y^3 = 0 \\ 3x^2 + y^2 = 0 \end{cases}$$

Supply 0 for a coefficient to the missing powers of x.

II. $$\begin{cases} m^2x - 2mx^2 + 1 = 0 \\ m + x^2 - 3mx = 0 \end{cases}$$ Eliminate m.

III. $$\begin{cases} ax^2 + bx + c = 0 \\ x^2 + qx + r = 0 \end{cases}$$ Eliminate x.

IV. $\left\{\begin{aligned} ax^2 + bx + c &= 0 \\ x^3 - 1 &= 0 \end{aligned}\right.$ Eliminate x.

V. $\left\{\begin{aligned} a\lambda^4 - \lambda^2 - 1 &= 0 \\ \lambda^3 - k\lambda &= 4 \end{aligned}\right.$

Determine k so that these equations in λ may be consistent.

PART II.

Art. 4.—Trace the curves

$$y = x^3; \qquad y = 2x^3 - 4x^2 + 1;$$
$$y = 3x; \qquad y = x^4 - 4x^3 + 11x^2 - 9x + 10.$$

Art. 6.—Find the values of $7x^5 - 9x^2 + 2$ when $x = 3, \frac{1}{2}, 6, 10$. Of $8x^4 - 9x^3 + 1$ when $x = 0, 3, 5, 6, 8$.

Art. 9.—What are the moduli of $3 + 4i$, $3 - 4i$, $7 - 10i$, -9, $-3i$, $16 - 25i$?

Art. 27.—Put $3 + 5i$ in the form $r(\cos\vartheta + i\sin\vartheta)$.

Ans. $r = \sqrt{34}$; $\vartheta = \tan^{-1}\frac{5}{3}$. Hence $\log\tan\vartheta = \log\frac{5}{3} = 0.2203$; consequently $\vartheta = 58°\ 56' = 58°.933$. Now we know that the angular unit $= 57°.295$; hence $\vartheta = \dfrac{58.933}{57.295} = 1.026$ angular units.

Put in the same form $6 - 2i$; $5 - 4i$; $8 - 11i$.

Art. 28.—Put in the exponential form:

$3 + 5i$; (*Ans.* $\sqrt{34}e^{1.026i}$;) $7 + 2i$; $6i$; 7; $-11 - 2i$.

Art. 33.—Find a cube root of $3 + 5i$.

Ans. $\sqrt[6]{34}e^{\frac{1.026}{3}i} = \sqrt[6]{34}e^{0.342i} = \sqrt[6]{34}\{\cos 19°\ 35' + i\sin 19°\ 35'\}$.

Otherwise

$$3 + 5i = \sqrt{34}(\cos 58°\ 56' + i\sin 58°\ 56');$$

hence

$$(3+5i)^{\frac{1}{3}} = \sqrt[6]{34}\left(\cos\frac{58^\circ\ 56'}{3} + i\sin\frac{58^\circ\ 56'}{3}\right)$$
$$= \sqrt[6]{34}\,(\cos 19^\circ\ 35' + i\sin 19^\circ\ 35').$$

Find a fourth root of $2+8i$; a fifth root of $11-6i$.

Arts. 35, 36.—Solve $x^3 = 1$. Denote its roots by $1, a, b$. Show that $b = a^2$; hence the roots are $1, a, a^2$. Prove that $1 + a + a^2 = 0$, and that $\begin{vmatrix} 1-a & a^2 \\ -a & a^2 & 1 \\ a^2 & 1-a \end{vmatrix} = 4$, while

$$\begin{vmatrix} 1 & a & a^2 & 1 \\ a & a^2 & 1 & 1 \\ a^2 & 1 & 1 & a \\ 1 & 1 & a & a^2 \end{vmatrix} = 3\sqrt{-3}.$$

(Wentworth.)

Solve $x^4 = 1$; $x^4 = -1$; $x^6 = 2$.

Arts. 42, 43, 45.—Given the following sets of roots, form the equations which they satisfy:

$1, 2, 3$; $1-i, 1+i, 2, 4$; $3-2i, 3+2i, 7, 1$; $-i, +i, 0, 1$.

Art. 49.—Form the sums of the squares, cubes, etc., of the roots of

$$x^3 = 1; \qquad x^3 - x + 1 = 0; \qquad x^3 - x^2 + 2x - 1 = 0;$$
$$x^4 - 4x^3 + 2x^2 - x + 1 = 0.$$

Arts. 50, 51.—From the equation whose roots are the reciprocals of those of $x^3 = 1$; $x^3 = -1$; $x^3 - 8x + 1 = 0$; $x^3 - 11x + 4 = 0$; $x^6 - 4x^4 + 11x^3 - x^2 + x - 9 = 0$; and compute the sum of their squares, cubes, etc.

Art. 52.—Express in terms of the coefficients:

$$\Sigma\alpha,\ \Sigma\alpha^3,\ \Sigma\alpha^4,\ \Sigma\alpha^2\beta\gamma,\ \Sigma\alpha^2\beta^2\gamma,\ \Sigma\alpha^2\beta\gamma\delta$$

for the equation $\quad x^n + bx^{n-1} + cx^{n-2} + \ldots = 0.$

Arts. 57, 60.—Compute by symmetric functions and by the dialytic method the resultants of:

$$\text{I.}\ \begin{cases} x^3 - 7x^2 + 4 = 0 \\ x^2 - 5x - 6 = 0 \end{cases} \qquad \text{II.}\ \begin{cases} x^4 - 1 = 0 \\ x^2 - 2x + 1 = 0 \end{cases}$$

$$\text{III.}\ \begin{cases} x^3 - x^2 + x - 1 = 0 \\ x^2 - 4x + 4 \quad = 0 \end{cases}$$

PART III.

Art. 17.—Put the following equations in the form

$$x^n + p_1 x^{n-1} + p_2 x^{n-2} + \ldots = 0,$$

where $p_1, p_2, \ldots$ are whole numbers:

$$3x^3 - \tfrac{1}{2}x^2 + \tfrac{1}{11} = 0; \quad 5x^4 - \tfrac{1}{3}x = 1; \quad 11x^6 - \tfrac{1}{7}x = 4.$$

Art. 24.—Compute to five places the roots of

$x^4 - 4x^3 - 43x^2 - 58x + 240 = 0;$
$x^3 - 12x^2 - 43x - 30 = 0;$
$x^5 - x^4 - 187x^3 - 359x^2 + 186x + 360 = 0.$ (Wentworth.)

How many real roots has each equation? How many positive roots? how many negative?

Compute to seven places the two nearly equal roots of

$$x^3 + 11x^2 - 102x + 181 = 0.$$

THE END.